Fractional Hadamard formula and applications

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Abstract

In this talk, we are interested in finding the boundary expression of the one-sided shape derivative of the best constant in the fractional Sobolev inequality. The Fractional Laplace operator $(-\Delta)^s$, $s \in (0,1)$ has attracted extensive attention in the last few decades due to its appearance in mathematical model describing phenomenon ranging from Quantum mechanics, Probability, Finance, Biology etc. For instance, in Quantum mechanics, it is used to describe random walks of particles with arbitrary long jumps. In probabilistic point of view, the fractional Laplace operator appears as infinitesimal generator of Lévy processes. It can be pointwisely defined, when acted on sufficiently regular functions, by the principal value integral

$$(-\Delta)^s u(x) = P.V\, c_{n,s} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy, \qquad \text{for all } s \in (0,1) \text{ and all } x \in \mathbb{R}^n$$

where $c_{n,s} > 0$ is some normalizing constant. The fractional Sobolev inequality states that, for any bounded open set $\Omega \subset \mathbb{R}^n$, there exists a constant $C = C(n,s,\Omega,p)$ such that

$$\|u\|_{L^p(\Omega)} \leq C \int_{\Omega} u(-\Delta)^s u\, dx, \qquad \text{for all } u \in C_c^\infty(\Omega) \text{ and for all } 1 \leq p \leq \frac{2n}{n-2s}.$$

The inverse of the **best constant** in the estimate above is given by

$$\lambda(\Omega) := \inf_{u \in C_c^\infty(\Omega)} \left\{ \int_{\Omega} u(-\Delta)^s u\, dx, \int_{\Omega} |u|^p dx = 1 \right\}$$

We consider the functional $\Omega \mapsto \lambda(\Omega)$ and computed the "boundary expression" of its one-sided shape derivative. In particular, we obtained the "nonlocal" version of the classical Hadamard variational formula for the first Dirichlet fractional eigenvalue. As an application, we consider the maximization of the first and second Dirichlet fractional eigenvalues in simply connected domains bounded by two spheres and prove that the maximum is attained in both cases by concentric spheres.

The talk is based on three papers: "A fractional Hadamard formula and applications" joint work with M.M. Fall (African Institut for Mathematical Sciences in Senegal) and Tobias. Weth (Goethe University, Frankfurt am Main); "Symmetry of odd solutions to equations with the fractional Laplacian" joint work with Sven Jarohs (Goethe University, Frankfurt am Main) and "Nonradiality of second fractional eigenfunctions of thin annuli", joint work with Sven Jarohs.