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The noncommutative geometry of matrix polynomial algebras

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In noncommutative affine algebraic geometry, representation spaces of not necessarily commutative algebras are the main objects of study. Here, various more or less specialized matrix subalgebras of full matrix algebras over an ambient polynomial ring show up naturally as interesting classes of algebras. One can name Cayley-Hamilton algebras of degree $n \in \mathbb{N}$ arising from the study of Gl_n-varieties by [7], and abstractly defined as algebras A admitting a formal trace map $tr_A : A \longrightarrow \mathrm{Z}(A)$ satisfying the Cayley-Hamilton identity. Here, each affine free Cayley-Hamilton algebra of degree n is a specialized matrix polynomial algebra in commuting variables, called a *trace algebra* [7, § 1.4; § 1.8, Thm 1.16]. Next in the realm of noncommutative deformation theory as introduced by Laudal and Eriksen [5, 6, 3] and further refined by Siqveland [10, §5], the authors obtained a noncommutative affine scheme structure on the irreducible representations of affine algebras over an algebraically closed field, allowing to reconstruct the algebras in some cases. Here free matrix polynomial algebras and their completions showed up as important tools in the computation of pro-representing hulls (the so-called formal moduli) [3, §5,p 105].

In this work we are essentially concerned with the geometry of matrix polynomial extensions of a coefficient algebra by elementary matrix-variables, of which we have formally introduced and started investigating the structure theory in [8], and whose definition we now recall. Let $\Bbbk$ be a base commutative ring, R a coefficient $\Bbbk$-algebra and consider an ambient polynomial ring extension $R\langle \boldsymbol{X}; \boldsymbol{C}\rangle$ of R by a set of non-necessarily commuting variables, subject to some commutativity relations '$xy = yx$' with (x, y) running a prescribed subset $\boldsymbol{C} \subset \boldsymbol{X} \times \boldsymbol{X}$. Let a positive natural number $n \in N$ be fixed, write $\mathbb{E} = \{\boldsymbol{e}_{i,j} : 1 \leq i, j \leq n\}$ for the canonical $\Bbbk$-basis of the full matrix ring $\mathcal{M}_n(\Bbbk)$ and consider a family $\mathbb{X} = \bigcup_{1 \leq i,j \leq n} \mathbb{X}_{i,j}\boldsymbol{e}_{i,j} \subset \{x\boldsymbol{e}_{i,j} : x \in \boldsymbol{X}, 1 \leq i, j \leq n\}$ of elementary matrix-variables, to which one may also adjoin a multiplicatively closed subset $\boldsymbol{E} \subset \mathbb{E}$ including all the idempotent elementary matrices. Then the *matrix polynomial algebra extension* of R by $(\boldsymbol{E}, \mathbb{X}, \boldsymbol{C})$ is the subalgebra

$$R\langle \boldsymbol{E}, \mathbb{X}; \boldsymbol{C}\rangle = R\langle \boldsymbol{e}, X \ : \ \boldsymbol{e} \in \boldsymbol{E}, X \in \mathbb{X}; \boldsymbol{C}\rangle$$

*

of the full matrix algebra $\mathcal{M}_n(\boldsymbol{R})$, generated over $R \cdot \boldsymbol{E} = \oplus_{\boldsymbol{e} \in \boldsymbol{E}} R\boldsymbol{e}$ by the elementary matrix-variables in $\mathbb{X}$.

A description of the main contributions of this work follows. At first glance, an in-depth re-visitation of the spectrum of maximal left (right, or bilateral) ideals of general matrix algebras $A = (A_{i,j})_{1 \le i,j \le n}$ is necessary, allowing us to clarify some recent imprecise statements appearing in the algebraic geometry of (n-pointed) matrix algebras by [9, § 3, Prop 5], [10, Def 9]. This leads to a precise description of the spectrum $\mathrm{Sp}(A)$ of maximal left (or right) ideals and of the scheme $\mathrm{Irr}A$ of irreducible representations of A in terms of those of the *diagonal algebra* $\mathrm{diag}(A) = \oplus_{i=1}^{n} A_i$. Essentially, the irreducible representations of A are induced by those of $\mathrm{diag}(A)$ and the associated geometric induction map $f : \mathrm{Irrdiag}(A) \longrightarrow \mathrm{Irr}A$ is a continuous surjection of Zariski spaces, with injective restrictions on each $\mathrm{Irr}A_k$ for $1 \le k \le n$. This means that the Zariski space $\mathrm{Irr}A$ is obtained by gluing together in a natural way the Zariski spaces $\mathrm{Irr}A_k$, $1 \le k \le n$.

Secondly specializing to noncommutative affine geometry of matrix polynomial algebras, an important step we achieve is concerned with a generalization of the Amitsur-Small Nullstellensatz [1, 4, Thm 17.6] to matrix polynomial extensions $\mathbf{A}$ in commuting variables of an extension of a simple artinian algebra $\mathfrak{L}$ by an *essentially finitely generated* commutative monoid. This critical result for geometry says that every irreducible representation of $\mathbf{A}$ is a left $\mathfrak{L}$-module with finite length. And when $\mathfrak{L}$ is finite-dimensional over its center $\mathbb{K}$, we are able to deduce that the primitive spectrum $\mathsf{PrimSpec}(\mathbf{A})$ coincides with the spectrum $\mathsf{Max}(\mathbf{A})$, and the primitive quotients of $\mathbf{A}$ are full matrix rings over finite-dimensional division $\mathbb{K}$-algebras while the isoclasses of irreducible representations of $\mathbf{A}$ correspond bijectively to maximal ideals in $\mathbf{A}$. Next from our algebraic Nullstellensatz and the description of the scheme $\mathrm{Irr}\mathbf{A}$ in terms of the schemes $\mathrm{Irr}\mathbf{A}_i$ for $1 \le i \le n$, we obtain a geometric form of Hilbert's Nullstellensatz (Theorem **??**).

For every $\Lambda \subset [\![1, n]\!]$, the associated *diagonal component* of $\mathbf{A}$ is the subalgebra $\mathbf{A}_\Lambda$ of the ambient polynomial ring generated by $\cup_{i \in \Lambda} \mathbf{A}_i$. Then as first application to nonnoetherian commutative geometry, we show that the (generally nonnoetherian) commutative monoid ring $\mathbf{A}_\Lambda$ is a Jacobson ring whose nonnoetherian commutative geometry is efficiently described by the geometry of an affine *essential subextension.* We equally characterize those matrix polynomial algebras in commuting variables which are geometric algebras (in the sense of [6]), reconstructible as algebras of observables from the scheme of irreducible representations.

Another application to nonnoetherian commutative algebraic geometry in the perspective of non-local algebraic geometry as underlined by [2] through a concept of *depiction.* Our main interest here addresses for monoid rings R an open question by [2, Quest 3.17]. Let $\boldsymbol{S} = \Bbbk[\mathsf{S}]$ and $\boldsymbol{R} = \Bbbk[\mathsf{T}]$ for a finitely generated commutative cancellative monoid S with $1 \notin \mathsf{S} \cdot (\mathsf{S} \backslash \{1\})$ and T a submonoid in S. Our main contributions in this direction (Theorem **??** and Corollary **??**) summarize as it follows.

❍ If the monoid ring $\boldsymbol{R}$ is nonnoetherian, then there exists $\mathfrak{q} \in \mathsf{Spec}\,\boldsymbol{S} \backslash \mathsf{Max}\,\boldsymbol{S}$ with $\mathfrak{q} \cap \boldsymbol{R} \in \mathsf{Max}(\boldsymbol{R})$. And whenever $\boldsymbol{S}$ is a depiction of $\boldsymbol{R}$, the ring $\boldsymbol{R}$ is nonnoetherian if and only if it admits a closed point of positive geometric dimension.

❍ There are general conditions on the shape of the submonoid T ensuring that $\boldsymbol{R}$ admits a depiction $\Bbbk[S']$ with $S' \subset S$.

❍ Let $\Lambda \subset [\![1, n]\!]$ and assume that $\mathbb{X}$ is finite. Whenever $\mathbb{X}$ has pairwise disjoint components (or has a more general appropriate shape), the algebra $\mathbf{A}_\Lambda$ admits a depiction and, $\mathbf{A}_\Lambda$ is nonnoetherian if and only if it admits a closed point with positive geometric dimension.

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