An introduction of amœbas in Mathematics

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Abstract

The notion of amoeba played a significant role in the development of tropical geometry. It led also to many important applications for example in asymptotic theory, thermodynamics, in *dimers*[1] theory or more generally in computational biology. Such a notion appeared in the end of XX^{e} century[2] as a useful tool towards a quick visualization of algebraic hypersurfaces (one can profit of the concavity of the logarithm function on $\mathbb{R}^{+*}$ and hence its slow growth towards $+\infty$). The terminology *amœba* arises (we will soon understand why) from biology. The definition of the *amœba of an algebraic hypersurface* appeared for the first time around 1990 in the famous treatise by I. N. Gelfand, M. Krapanov and A. Zelevinsky [19].

Amoeba and coamœba of an algebraic hypersurface in $\mathbb{T}^n$

Definition 0.0.1 (archimedean amœba of an algebraic hypersurface in $\mathbb{T}^n$) Let $\boldsymbol{Z}_{\mathbb{T}^n}(f)$ be an algebraic hypersurface of the complex torus $\mathbb{T}^n = \text{Spec}(\mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}])$ defined as the support $f^{-1}(\{0\})$ of $\text{div}_{\mathbb{T}^n}(f)$, where f is a Laurent polynomials in n complex variables with constant coefficients. The *archimedean amœba* [3] is be definition the subset[4]

$$\mathscr{A}_f := \text{Log}\left(\boldsymbol{Z}_{\mathbb{T}^n}(f)\right) \tag{0.0.1}$$

[1]A *dimer* is a molecular complex of chemicals with two unities (as *monomer* when there is a single unity, *trimer* when there are three unities, *oligomer* when the number of unities is finite, *polymer* when it is infinite...)

[2]In fact one could guess the genesis of the concept goes back to the mathematical contribution of Isaac Newton, 1643-1727 and more recently to that of Victor Puiseux, 1820-1883.

[3]One precise here the terminology *archimedean* since the absolute value $|\ |$ which is involved here arises from the archimedean absolute value on $\mathbb{Q}$. If $f = \sum_{\alpha \in \text{Supp}(f)} c_\alpha X^\alpha$ happens for example to be defined over $\mathbb{Q}$ (that is $c_\alpha \in \mathbb{Q}^*$ for any $\alpha \in \text{Supp}(f)$), then the *non-archimedean amœba* of f with respect to the p-adic ultrametric absolute value $|\ |_p$ (when p is a prime integer) is the support (no need here to use the morphism Log anymore since we are already in the tropical setting) in $\mathbb{R}^n = (\text{Trop} \setminus \{-\infty\})^n$ of the tropical cycle $\text{div}_{\text{Trop}}(\mathfrak{f})$, where

$$\mathfrak{f} = \boxplus_{j=1}^{N} \log |c_\alpha|_p \boxtimes X_1^{\boxtimes^{\alpha[j],1}} \boxtimes \cdots \boxtimes X_n^{\boxtimes^{\alpha[j],n}}.$$

[4]One uses sometimes instead of Log the map $-\text{Log}$ since $-\text{Log}$ is the usual *tropicalization map* in the archimedean setting (note that $-\text{Log}$ was already used for example in Definition **??** for tropical currents). We will keep here to the use of Log since it is more familiar to complex analysts.

of $\mathbb{R}^n$. Note that it depends only on the support $f^{-1}(\{0\})$ of $\mathrm{div}_{\mathbb{T}^n}(f)$ despite of the terminology (archimedean amœba of f), that is multiplicities are not taken into account[5].

Example 0.0.1 (amœba of an affine line in $\mathbb{T}^2$) We follow here the presentation in [38]. Consider the affine line $L_0 := \{z_1 + z_2 - 1 = 0\}$ in $\mathbb{T}^2$. A point (z_1, z_2) in $\mathbb{T}^2$ belongs to L_0 if and only if $z_1 \in \mathbb{T}$ and $z_2 \in \mathbb{T}$ satisfy the three conditions:

$$|z_1| + |z_2| \geq 1, \quad |z_1| + 1 \geq |z_2|, \quad |z_2| + 1 \geq |z_1|. \tag{0.0.2}$$

This amounts to say that the family $\{1, |z_1|, |z_2|\}$ is *not lobsided*[6] in the archimedean sense. Such conditions (0.0.2) stand for the necessary and sufficient conditions ensuring that the positive numbers $1,\ |z_1|,\ |z_2|$ can be interpreted as the lengths of the 1-dimensional faces (facets) of a triangle. Then, the amœba of the complex line $\{z_1 + z_2 = -1\}$ is the image of the domain

$$\big\{(u,v) \in \,]0, \infty[^2\,; u + v \geq 1,\ u + 1 \geq v,\ v + 1 \geq u\big\}$$

by the map $(u, v) \mapsto (\log u, \log v)$.

Remark 0.0.1 (closedness of $\mathscr{A}_f$) Since $\mathbb{T}^n \to \mathbb{R}^n$ is continuous map, the archimedean amoeba of $f \in \mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$ is a closed subset in $\mathbb{R}^n$. Note also that the open set $\mathrm{Log}^{-1}(\mathbb{R}^n \setminus \mathscr{A}_f) \subset \mathbb{T}^n$ is a Reinhardt open subset (that is an open subset which is invariant under the multiplicative action on $\mathbb{T}^n$ of the real torus $\mathbb{T}^n_{\mathbb{R}}$).

We note that the Log map involved in the definition of the archimedean amœba is the real part of the complex (multivalued if one considers it from $\mathbb{T}^n$ to $\mathbb{C}^n$) holomorphic logarithm

$$\log : z \in \mathbb{T}^n \mapsto \mathrm{Log}\,(z) + i \arg\,(\mathrm{z}) \in \mathbb{R}^n + i\,\mathbb{T}^n_{\mathbb{R}}.$$

[5]We will keep the notation $\mathscr{A}_f$ instead of $\mathscr{A}_{\boldsymbol{Z}_{\mathbb{T}^n}(f)}$ for the sake of simplicity.

[6]A finite family of srictly positive numbers is said to be *lobsided* in the archimedean sense if and only if one of its elements is strictly bigger than the sum of the others.

The archimedean amœba of the Laurent polynomial f is therefore the projection of $\log \boldsymbol{Z}_{\mathbb{T}}(f) \in \mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$ on $\mathbb{R}^n$ (taking the real part). One could as well consider the projection of $\log \boldsymbol{Z}_{\mathbb{T}^n}(f)$ on $\mathbb{T}^n_{\mathbb{R}}$ (taking this time the imaginary part). Such projection is called the *coamœba*[7] of the Laurent polynomial f and denoted as $\mathrm{co}\mathscr{A}_f$.

Archimedean amoebas and Laurent series

Let $f \in \mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$ be a Laurent polynomial with complex coefficients, that is a regular algebraic function on $\mathbb{T}^n = \mathrm{Spec}(\mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$, which is *truly* in n complex variables[8]. The domain of holomorphy of the rational function $1/f$ consists in the union of the pre-images *via* Log of the open connected components E of the open subset $\mathbb{R}^n \setminus \mathscr{A}_f$. In each $\mathrm{Log}^{-1}(E)$ (which is a Reinhardt domain), the rational function $\zeta \mapsto 1/f(\zeta)$ can be developped as a Laurent series $\sum_{k\in\mathbb{N}} a_{E,k}\zeta^{\nu_{E,k}}$, where $a_{E,k} \in \mathbb{C}$ and $\nu_{E,k} \in \mathbb{Z}^n$.

An important fact is that *there is a one-to-one correspondence between the set of connected components of* $\mathbb{R}^n \setminus \mathscr{A}_f$ *and possible convergent Laurent developments (with precisely domains of convergence the Reinhardt open subsets* $\mathrm{Log}^{-1}(E)$ *when E is an open connected component of* $\mathbb{R}^n \setminus \mathscr{A}_f$*) for the rational function* $z \in \mathbb{T}^n \mapsto 1/f(z)$ [17].

On the other side, *there is a one-to-one correspondence between the pre-images via* arg *of the open connected components* $\mathfrak{E}$ *of* $\mathbb{T}^n_{\mathbb{R}} \setminus \overline{\mathrm{co}\mathscr{A}_f}$ *(the coamœba being not closed in* $\mathbb{T}^n$ *as the amœba is in* $\mathbb{R}^n$*) and possible integral Mellin representations for such rational function* $z \mapsto 1/f(z)$ *(with convergence domains precisely the* $\arg^{-1}(\mathfrak{E})$) [29].

Both such correspondences could be indeed of interest in systems theory or image processing. We will concentrate in this thesis on the first correspondence ((components C of $\mathbb{R}^n \setminus \mathscr{A}_f$) $\leftrightarrow$ (Laurent developments of $1/f$) since we will not explore further here the concept of coamœba.

The rational function $z \in \mathbb{T}^n \mapsto 1/f(z)$ admits in $\mathrm{Log}^{-1}(E)$ (E being a selected connected component of $\mathbb{R}^n \setminus \mathscr{A}_f$) the convergent Laurent development

$$\frac{1}{f(z)} = \sum_{k=0}^{\infty} a_k z^{\nu_k}, \quad a_k \in \mathbb{C}, \tag{0.0.3}$$

where the complex coefficients $a_k = a_{E,k}$ and the exponents $\nu_k = \nu_{E,k} \in \mathbb{Z}^n$ are given by the integral formulas

$$a_k = \frac{1}{(2i\pi)^n} \int_{\mathrm{Log}^{-1}(\{x\})} \frac{1}{f(z)} z^{-\nu_k} \frac{dz}{z}, \quad \text{where} \quad \frac{dz}{z} = \frac{dz_1}{z_1} \wedge ... \wedge \frac{dz_n}{z_n}, \tag{0.0.4}$$

where x denotes an arbitrary point in E and the real n-dimensional torus $\mathrm{Log}^{-1}(\{x\})$ with support in $\mathbb{T}^n \setminus \boldsymbol{Z}_{\mathbb{T}^n}(f)$ is oriented such the n-differential form $d\arg(z_1) \wedge \cdots \wedge d\arg(z_n)$ is a volume form on its support. The fact that a_k does not depend on the choice of x comes from the fact that

[7] It remains unclear what could be the companion of the concept of coamœba in the ultrametric (non-archimedean) setting when $f \in \mathbb{Z}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$. Therefore, we do not here to precise that we are in the archimedean setting.

[8] This means that its Newton polyhedron $\Delta(f)$ (namely the closed convex envelope of its support) is n-dimensional. If it is not the case, it means, up to a $\mathbb{Q}$-rational linear change of coordinates and a rescaling, that the situation reduces to that where f can be considered as a Laurent polynomial in strictly less than n variables.

$\mathrm{Log}^{-1}(\{x\})$ and $\mathrm{Log}^{-1}(\{y\})$ are homologous when x and y lie in both in E. Multi-exponents $\nu_k \in \mathbb{Z}^n$ such as $a_k \neq 0$ can then be organized as indexed by $\mathbb{N}$, so that the Laurent series on the right-hand side of (0.0.3) converges normally on any compact subset of $\mathrm{Log}^{-1}(E)$.

Remark 0.0.2 If $x = (x_1, ..., x_n)$ is fixed in E, then (0.0.3) can be reformulated as

$$\frac{1}{f(e^{x_1+i\theta_1}, ..., e^{x_n+i\theta_n})} = \sum_{k=0}^{\infty} a_k \, e^{\langle \nu_k, x\rangle} \, e^{i\langle \nu_k, \theta\rangle}$$

and the integral expression for the coefficients a_k is just Fourier inversion formula.

Let us state here the complete result from [17] (completed here with statements from [30]) with respect to this question about Laurent developments for $1/f$ as well as related ones.

Theorem 0.0.1 *Each connected component E of $\mathbb{R}^n \setminus \mathscr{A}_f$ is convex and there is a one-to-one correspondence between the set of connected components of the complement set $\mathbb{R}^n \setminus \mathscr{A}_f$ and possible convergent Laurent developments $1/f(z) = \sum_{k\in\mathbb{N}} a_k z^{\nu_k}$ (with precisely domains of convergence the Reinhardt open subsets $\mathrm{Log}^{-1}(E)$ when E is an open connected component of $\mathbb{R}^n \setminus \mathscr{A}_f$) for the rational function $z \in \mathbb{T}^n \mapsto 1/f(z)$.*

Remark 0.0.3 The convexity of each E follows from the fact that the Reinhardt domain of convergence of a Laurent series $\sum_{k\in\mathbb{N}} a_k z^{\nu_k}$ is always logarithmically convex in $\mathbb{T}^n$, that is of the form $\mathrm{Log}^{-1}(U)$, where U is an open convex subset of $\mathbb{R}^n$.

The Ronkin function R_f and its gradient

Ronkin function is the continiuous function define as :

$$x \in \mathbb{R}^n \longmapsto \int_{\mathbb{T}^n_{\mathbb{R}}} \log |s(e^{x_1+i\theta_1}, ..., e^{x_n+i\theta_n})|_{\psi} \, d\nu_{\mathbb{T}^n_{\mathbb{R}}}(\theta). \tag{0.0.5}$$

Let $f \in \mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}$ a Laurent polynomial with complex coefficients which is truly in n variables ($\dim \Delta_f = n$). As we have already pointed it out, the archimedean amœba $\mathscr{A}_f$ (that is the support of the $(1,1)$-super-current $dd^c \boldsymbol{R}_f$) does not depend on f, but in fact only on $\boldsymbol{Z}_f = f^{-1}(0)$. On the opposite, the Ronkin function R_f does. It carries indeed the algebraic information (about multiplicities) that the archimedean $\mathscr{A}_f$ does not reflect.

Precisely, one can attach to each open (convex) component connected component E of $\mathbb{R}^n \setminus \mathscr{A}_f$ a *multiplicity* $\mu_E \in \Delta_f \cap (\mathbb{Z}^n)^\star$. To see that, fix a connected component E of $\mathbb{R}^n \setminus \mathscr{A}_f$. Let $x \in E$ and $z_x = (z_{x,1}, ..., z_{x,n})$ in the fiber $\mathrm{Log}^{-1}(\{x\})$. Then, for any $j = 1, ..., n$,

$$\mu_{E,j}(x) = \int_{\mathbb{T}_{\mathbb{R}}} \Big(\frac{f'_{z_j}}{f}\Big)\big(z_{x,1}, ..., z_{x,j-1}, e^{x_j+i\theta}, z_{x,j+1}, ..., z_{x,n}\big) \, e^{x_j+i\theta} \, d\nu_{\mathbb{T}_{\mathbb{R}}}(\theta) \in \mathbb{Z} \tag{0.0.6}$$

as a consequence of the argument principle. An homotopy argument asserts that this integer is in fact independent on the choice of x, once x remains in the connected component E.

Definition 0.0.2 (multiplicity of a connected component of $\mathbb{R}^n \setminus \mathscr{A}_f$) The *multiplicity* $\mu_E \in (\mathbb{Z}^n)^\star$ of the connected component E of $\mathbb{R}^n \setminus \mathscr{A}_f$ is the element in $(\mathbb{Z}^n)^\star$ which component with index j is the integer $\alpha_{E,j}(x)$ for $x \in E$, that is, from the topological point of view, the degree of the loop

$$\theta \in \mathbb{T}_{\mathbb{R}} \longmapsto \big(z_{x,1}, \dots, z_{x,j-1}, e^{x_j + i\theta}, z_{x,j+1}, \dots, z_{x,n}\big)$$

for $z_x \in \mathrm{Log}^{-1}(\{x\})$, this degree being independent on the choice of x in E.

Let us quote here the fundamental result from [17].

Theorem 0.0.2 *For each connected component E of $\mathbb{R}^n \setminus \mathscr{A}_f$, the multiplicity μ_E belongs to $\Delta_f \cap (\mathbb{Z}^n)^\star$. Moreover the map $\mathrm{MULT}_f : E \mapsto \mu_E$ is injective, in general non surjective, although its image contains the vertices of Δ_f. One has then*

$$\#\{\tau \prec \Delta_f\,;\, \dim \tau = 0\} \leq \dim H_0(\mathbb{R}^n \setminus \mathscr{A}_f) \leq \#(\Delta_f \cap (\mathbb{Z}^n)^\star). \qquad (0.0.7)$$

Definitions 0.0.1 (optimality, rigidity, genus of the archimedean amœba $\mathscr{A}_f$) *Let Δ be a n-dimensional $(\mathbb{Z}^n)^\star$-integer polyhedron.*

1. The archimedean amœba $\mathscr{A}_f$ of a Laurent polynomial with Newton polyhedron Δ is said to be *optimal*[9] if the number of connected components for $\mathbb{R}^n \setminus \mathscr{A}_f$ is *maximal*, that is equal to $\#(\Delta \cap (\mathbb{Z}^n)^\star)$.

2. The archimedean amœba $\mathscr{A}_f$ of a Laurent polynomial with Newton polyhedron Δ is said to be *solid*[10] if the number of connected components for $\mathbb{R}^n \setminus \mathscr{A}_f$ is *minimal*, that is equal to the number of vertices in Δ.

3. The defect between the number of connected components of $\mathbb{R}^n \setminus \mathscr{A}_f$ and the number of vertices of Δ is the *topological genus* of $\mathscr{A}_f$ (that is the number of bounded connected components of $\mathbb{R}^n \setminus \mathscr{A}_f$).

One has the following important result [30].

Theorem 0.0.3 *Let f be a Laurent polynomial in n variables with complex coefficients such that $\dim \Delta_f = n$. For any $\alpha \in \Delta_f \cap (\mathbb{Z}^n)^\star$, let the polyhedron be $\tau_\alpha \prec \Delta_f$ be*

- *either the 0-dimensional face $\{\alpha\}$ of Δ_f when α is a vertex of Δ_f ;*

- *either the unique face τ of Δ_f which admits α in its relative interior (that is its interior in the affine $\mathbb{R}$-subspace A_τ)*[11].

If there exists a connected component E of $\mathbb{R}^n \setminus \mathscr{A}_f$ such that $\mathrm{MULT}_f(E) = \mu$, the open convex set E admits as recession cone[12] *the $(n - \dim \tau_\mu)$-dimensional cone*

$$\mathrm{pol}_{\tau_\mu \prec \Delta_f} = \big\{x \in \mathbb{R}^n\,;\, \{\xi \in \Delta_f\,;\, \langle \xi, x\rangle = \max_{u \in \Delta_f} \langle u, x\rangle\} = \tau_\mu\big\}.$$

[9] For any Δ, there are optimal amœbas.

[10] The problem to decide wether $\mathscr{A}_f$ is solid as soon as the support of f coincides sur the set of vertices of Δ_f remains an open question.

[11] When α is interior to Δ_f, note that $\tau_\alpha = \Delta_f$.

[12] The *recession cone* of a convex open subset U of $\mathbb{R}^n$ is the largest strict cone σ in $\mathbb{R}^n$ such that $U + \sigma \subset U$ (in the sense of Minkowski addition of convex bodies) with respect to the ordering induced by the inclusion.

Coming back to the affine Ronkin function R_f, one observes that it is affine in any component E of $\mathbb{R}^n \setminus \mathscr{A}_f$. More precisely, one has the following result [30].

Theorem 0.0.4 (explication of R_f in $\mathbb{R}^n \setminus \mathscr{A}_f$) *Let $f = \sum_\alpha c_\alpha X^\alpha$ be a Laurent polynomial in n variables such that $\dim \Delta_f = n$. In each connected component E of $\mathbb{R}^n \setminus \mathscr{A}_f$, one has*

$$\forall\, x \in E, \quad R_f(x) = -\check{R}_f(\mu_E) + \langle \mu_E, x\rangle. \tag{0.0.8}$$

Let $1/f(z) = \sum_{k=0}^{\infty} a_{E,k}\, z^{E,k}$ be the convergent Laurent development of $1/f$ in the logarithmically convex Reinhardt open subset $\mathrm{Log}^{-1}(E)$ as in (0.0.3). For each vertex μ of Δ_f, the real number $-\check{R}_f(\mu)$ equals $\log|c_\mu|$. Finally, one has $\mathrm{stab}(R_f) = \Delta_f$.

Proof 0.0.1 The fact that R_f is C^∞ in the $\mathbb{R}^n \setminus \mathscr{A}_f$ follows from Lebesgue's differentiation theorem for integrals fonctions of a parameter. The explicitation of the gradient vector function $\vec{\nabla} R_f$ as a constant function equal to α_C in the connected component E of $\mathbb{R}^n \setminus \mathscr{A}_f$ follows from the application of this theorem together with the definition (0.0.6) of the components $\alpha_{E,j}$ of the multiplicity μ_E. Since a real convex function is the upper enveloppe of the affine functions which it dominates and R_f is affine in E, the fact that it can be expressed as (0.0.8) follows of the definition of the Legendre-Fenchel transform $\check{R}_f$. This concludes the proof of the first assertion of the theorem.
Let $\mu \in \Delta_f \cap (\mathbb{Z}^n)^\star$ be a vertex of Δ_f. Then it follows from theorem 0.0.3 that the recession cone of the component E such that $\mathrm{MULT}_f(C) = \mu$ equals the n-dimensional cone $\mathrm{pol}_{\{\mu\} \prec \Delta_f}$. Hence the convergent Laurent development of $1/f$ in the logarithmically convex Reinhardt open subset $\mathrm{Log}^{-1}(E)$ of $\mathbb{T}^n$ is

$$\frac{1}{f(z)} = \frac{z^{-\mu}}{c_\mu} \sum_{k=0}^{\infty} (-1)^k \Big(\sum_{\alpha \in \mathrm{Supp}(f) \setminus \{\mu\}} \frac{c_\alpha}{c\mu}\, z^{\alpha-\mu} \Big).$$

Then one has for any $z \in \mathrm{Log}^{-1}(E)$ that

$$\log|f(z)| = \log|c_\mu| + \langle \mu, \mathrm{Log}(z)\rangle + \log\Big| \sum_{k=0}^{\infty} (-1)^k \Big(\sum_{\alpha \in \mathrm{Supp}(f) \setminus \{\alpha\}} \frac{c_\beta}{c\mu}\, z^{\alpha-\mu} \Big) \Big|.$$

Averaging over the orbit of z with respect to the action of $\mathbb{T}^n_{\mathbb{R}}$ leads to

$$\int_{\mathbb{T}^n_{\mathbb{R}}} \log|f(z_1 e^{i\theta_1}, ..., z_n e^{i\theta_n})|\, d\nu_{\mathbb{T}^n_{\mathbb{R}}}(\theta) = \log|c_\mu| + \langle \mu\,,\, \mathrm{Log}(z)\rangle,$$

which shows that $-\check{R}_f(\mu) = \log|c_\mu|$. The final assertion about the stabilizer follows from the fact that the closures of the connected components E of $\mathbb{R}^n \setminus \mathscr{A}_f$ corresponding with the vertices of Δ_f have disjoint interiors and union equal to $\mathbb{R}^n$ ■

Contour of archimedean amœbas $\mathscr{A}_f$ and Gauss logarithmic map

Let $f \in \mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$ a Laurent polynomial in n variables such that $\dim \Delta_f = n$. Let us suppose that f is *reduced* or *squarefree*, which means that f factorizes in $\mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$ as a product of distinct irreducible Laurent polynomials.

Definition 0.0.3 (contour of the archimedean amœba $\mathscr{A}_f$) The *contour* of the archimedean amœba $\mathscr{A}_f$ is be definition the set of critical values of the restriction of the map Log to the algebraic hypersurface $\boldsymbol{Z}_f = f^{-1}(\{0\})$ of $\mathbb{T}^n$. It consists into two classes of points in $\mathbb{R}^n$, namely :

- the images of points $z \in \boldsymbol{Z}_f^{\rm sing}$;
- the images of critical points of the restriction of Log to $\boldsymbol{Z}_f^{\rm reg}$.

Let us suppose that f is reduced, that is squarefree i.e factorized as a product of distinct irreducible Laurent polynomials. The description of the contour of $\mathscr{A}_f$ lies on the so called *Gauss logarithmic map*. Let us explain its construction. If $z_0 \in \boldsymbol{Z}_f^{\rm reg}$, then the partial derivatives $\partial_{z_1} f(z_0)$, $\partial_{z_n} f(z_0)$ do not vanish simultaneously and

$$\gamma_f(z_0) = \Big[z_{0,1}\,\frac{\partial f}{\partial z_1}(z_0) : \cdots : z_{0,n}\,\frac{\partial f}{\partial z_n}(z_0)\Big] \in \mathbb{P}^{n-1}(\mathbb{C})$$

(when $n = 2$, $\mathbb{P}^1(\mathbb{C})$ being identified with the Riemann sphere $\mathbb{S} = \mathbb{S}^2$, one may consider then $\gamma_f(z_0)$ as a point in $\mathbb{S}$.

Definition 0.0.4 (Gauss logarithmic map of an algebraic hypersurface in $\mathbb{T}^n$) Let $f \in \mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$, reduced and such that $\dim \Delta_f = n$. The rational map $\gamma_f \;: \boldsymbol{Z}_f^{\rm reg} \to \mathbb{P}^{n-1}(\mathbb{C})$ *Gauss logarithmic map* of the reduced algebraic hypersurface $\boldsymbol{Z}_f$.

Remark 0.0.4 The Gauss logarithmic map extends as a rational map

$$\boldsymbol{\gamma}_f \;: \boldsymbol{Z}_f \longrightarrow \mathbb{P}^{n-1}(\mathbb{C})$$

which admits as polar set the singular set $\boldsymbol{Z}_f^{\rm sing}$.

The Gauss logarithmic map was originally introduced by Krapanov [21] for algebraic hypersurfaces in $\mathbb{T}^n$, but it can be extended naturally to $(n-m)$-dimensional *reduced algebraic complete intersections*, which means that $\boldsymbol{Z}_f = \boldsymbol{Z}_{f_1,...,f_m}$ is defined in $\mathbb{T}^n$ as the set of common zeroes of $m \in \{1, ..., n-1\}$ Laurent polynomials $f_1, ..., f_m$ in n variables such that the rank of the jacobian matrix $[\partial_{z_k} f_j(z)]_{1\leq j\leq m, 1\leq k\leq n}$ equals generically m on $\boldsymbol{Z}_f$. We also suppose that $\dim(\Delta_f) = n$, where $\Delta_f = \Delta(f_1) + \cdots + \Delta(f_m))n$, although this hypothesis is not essential (if it is not the case, the number n of variables can be lowered).

Let $\boldsymbol{Z}_f$ $(f = (f_1, ..., f_m))$ be such a reduced $(n-m)$ dimensional algebraic complete intersection in $\mathbb{T}^n$ with $\dim \Delta_f = n$. If $z_0 \in \boldsymbol{Z}_f^{\rm reg}$, the m lines of the matrix

$$\gamma_f(z_0) = \begin{pmatrix} z_{0,1}\,\dfrac{\partial f_1}{\partial z_1}(z_0) & \cdots & z_{0,n}\,\dfrac{\partial f_1}{\partial z_n}(z_0) \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ z_{0,1}\,\dfrac{\partial f_m}{\partial z_1}(z_0) & \cdots & z_{0,n}\,\dfrac{\partial f_m}{\partial z_n}(z_0) \end{pmatrix} \tag{0.0.9}$$

form a basis of the complex normal m-dimensional space (in $\log \mathbb{T}^n = \mathbb{R}^n + i\mathbb{R}^n = \mathbb{C}^n_w$) to the complex manifold $\log \boldsymbol{Z}_f$ about $\log z_0$ ($\log = \mathrm{Log} + i \arg$ is indeed a multivalued holomorphic function from $\mathbb{T}^n$ to $\mathbb{C}^n$, but one takes here any of its determinations). One can then consider $\gamma_f(z_0)$ ($f = (f_1, ..., f_m)$) represented as (0.0.9) as an element in the *Grassmanian complex manifold* $\mathtt{Gr}(m, n)$ whose elements are the *m-planes* (that is m-dimensional $\mathbb{C}$-subspaces) in $\mathbb{C}^n$ ($\mathtt{G}(1, n) \simeq \mathbb{P}^{n-1}(\mathbb{C})$ when $m = 1$).

Definition 0.0.5 (Gauss logarithmic map of reduced complete intersections in $\mathbb{T}^n$) Let $1 \leq m \leq n$. Let $f_1, ..., f_m$ be m elements in $\mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$ defining a reduced complete intersection in $\mathbb{T}^n$ and such that $\dim \Delta_f = n$. The *Gauss logarithmic map* $\gamma_f = \gamma_{f_1,...,f_m}$ is the rational map from $\boldsymbol{Z}_f^{\mathrm{reg}}$ to $\mathtt{G}(m, n)$ defined as (0.0.9).

Explicitation of the contour through the Gauss logarithmic map

Let $f \in \mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$ be reduced and such that $\dim \Delta_f = n$ (otherwise one can lower the number n of variables[13]). The following result [27, 26] precises the description of the contour $\mathrm{cont}(\mathscr{A}_f)$ of the archimedean amœba $\mathscr{A}_f$ in terms of the Gauss logarithmic map γ_f.

Theorem 0.0.5 (explicitation of the contour of amœbas of hypersurfaces) *Let $f \in \mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$ be a reduced Laurent polynomial truly in n variables (that is $\dim \Delta_f = n$). Then the contour of the amœba $\mathscr{A}_f$ is the closure in $\mathbb{R}^n$ of*

$$\mathrm{Log}\big(\gamma_f^{-1}(\mathbb{P}^{n-1}(\mathbb{R}))\big),$$

where the Gauss logarithmic map is considered from $\boldsymbol{Z}_f^{\mathrm{reg}}$ to $\mathbb{P}^{n-1}(\mathbb{C})$. It is also equal to $\mathrm{Log}(\boldsymbol{\gamma}_f^{-1}(\mathbb{P}^{n-1}(\mathbb{R}))$ is $\boldsymbol{\gamma}_f$ denotes the extension of γ_f as a rational map from $\boldsymbol{Z}_f$ to $\mathbb{P}^{n-1}(\mathbb{C})$.

A natural extension of this result to the case of reduced algebraic complete intersections in $\mathbb{T}^n$ was proposed by N. A. Bushueva and A. Tsikh in [1]. We will sketch here a proof of this result since this proof inspired some methods that will be described in chapter 5.

Theorem 0.0.6 (explicitation of the contour of algebraic reduced complete intersections) *Consider a vector $f = (f_1, ..., f_m)$ of elements in $\mathbb{C}[X_1^{\pm 1}, ..., X_n^{\pm 1}]$ which defines a reduced algebraic complete intersection $\boldsymbol{Z}_f$ with dimension $d = n - m$ in $\mathbb{T}^n$, with $\dim(\Delta_f) = n$. A point $z_0 \in \boldsymbol{Z}_f^{\mathrm{reg}}$ is a critical point for the restriction of Log to $\boldsymbol{Z}^{\mathrm{reg}}$ if and only $\gamma_f(z_0) \in \mathtt{Gr}(m, n)$ contains*

- *at least $n - 2d + 1$ linearly independent real vectors when $2d \leq n$;*
- *at least one real non zero vector when $2d > n$.*

Remark 0.0.5 (the two extreme cases $m = 1$ and $m = n - 1$)

1. When $m = 1$ ($\boldsymbol{Z}_f$ is an algebraic hypersurface in $\mathbb{T}^n$), $2d = 2(n-1)$ and $n - 2d + 1 = 3 - n$; since $2d \leq n \iff n \leq 2$, we recover in this case that the $\gamma(z_0) \in \mathtt{G}(1, n) = \mathbb{P}^{n-1}(\mathbb{C})$ contains one real non zero vector, which means that $\gamma(z) \in \mathbb{P}^{n-1}(\mathbb{R})$. As a consequence, one recovers Theorem 0.0.5 in this case.

[13] This last hypothesis is not an essential hypothesis, but we always assume it in order to avoid sometimes non-necessary discussions.

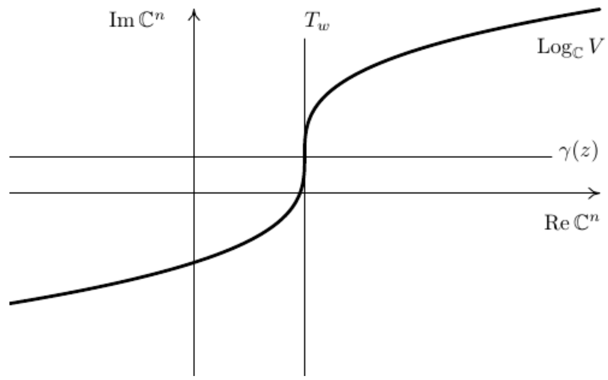

Figure 1

2. When $n \geq 2$ and $m = n-1$ ($\boldsymbol{Z}_f$ is then an algebraic curve defined as a reduced complete intersection in $\mathbb{T}^n$), $2d = 2$ and $n - 2d + 1 = n - 1$. We are in the first case, which means that all $(n-1, n-1)$-minors of the matrix $\gamma(z)$ are real.

Proof 0.0.2 Let $\pi_{\mathbb{R}}$ be the projection map from $\mathbb{R}^n + i\mathbb{R}^n = \mathbb{C}^n_w$ to $\mathbb{R}^n$ which associates to w its real part, so that $\mathrm{Log} = \pi_{\mathbb{R}} \circ \log$. Since log realizes as a multivalued holomorphic function a biholomorphism between $\mathbb{T}^n_z$ and $\mathbb{C}^n_w$, it transforms locally the d-dimensional manifold $\boldsymbol{Z}_f^{\mathrm{reg}}$ into a d-dimensional submanifold of $\mathbb{C}^n_w$. It is therefore the projection $\pi_{\mathbb{R}}$ from $\log \boldsymbol{Z}_f^{\mathrm{reg}}$ to $\mathbb{R}^n$ which is responsible for producing critical points for the restriction of Log to $\boldsymbol{Z}_f^{\mathrm{reg}}$. Given $w_0 = \log z_0 \in \log \boldsymbol{Z}_f^{\mathrm{reg}}$, let $T_{w_0}((\log \boldsymbol{Z}_f^{\mathrm{reg}})_{\mathbb{R}})$ be the $2d$-dimensional tangent plane[14] at w to the underlying $2d$-dimensional real manifold $(\log \boldsymbol{Z}_f^{\mathrm{reg}})_{\mathbb{R}}$ of the complex d-dimensional manifold $\log \boldsymbol{Z}_f^{\mathrm{reg}}$ about w_0. Consider the tangent map

$$\Pi_{w_0} = d(\pi_{\mathbb{R}})_{|(\log \boldsymbol{Z}_f^{\mathrm{reg}})_{\mathbb{R}}}(w_0) \ : T_{w_0}((\log \boldsymbol{Z}_f^{\mathrm{reg}})_{\mathbb{R}}) \longmapsto T_{\mathrm{Re}(w_0)}\mathbb{R}^n = \mathbb{R}^n.$$

The point z_0 is a critical point of $\mathrm{Log}_{\boldsymbol{Z}_f^{\mathrm{reg}}}$ if and only if we are in one of the two following situations :

- either $2d \leq n$ and in this case the tangent map $d(\pi_{\mathbb{R}})_{|(\log \boldsymbol{Z}_f^{\mathrm{reg}})_{\mathbb{R}}}(w_0)$ is non injective ;
- either $2d > n$ and in this case the tangent map $d(\pi_{\mathbb{R}})_{|(\log \boldsymbol{Z}_f^{\mathrm{reg}})_{\mathbb{R}}}(w_0)$ is non surjective.

Let us now point out that heuristically such condition of non-injectivity or non-surjectivity (depending whether $2d \leq n$ or $2d > n$) for the tangent map Π_{w_0} are connected with the property for a certain subspace of the (complex) normal subspace $\gamma(z_0)$ to $\log \boldsymbol{Z}_f^{\mathrm{reg}}$ at w_0 to be real. We refer to Figure 1 to visualize this : for w_0 to be a critical point for the restriction to $\pi_{\mathbb{R}}$ to $\log \boldsymbol{Z}_f^{\mathrm{reg}}$ about w_0, one needs such subpace to be *horizontal*, that is to have non imaginary component, which means *being real.*

Let us suppose for the sake of simplicity that $w_0 = 0$ and denote as T_0 the tangent plane at w_0 to $\log \boldsymbol{Z}_f^{\mathrm{reg}}$. Keeping in mind its complex structure of d-complex plane, it is defined by a system of equations

$$\langle \varphi_j + i\,\psi_j, u + iv \rangle = 0 \quad (j = 1, ..., n-d),$$

[14]Such a $2d$-dimension tangent space inherits a complex structure an thus can be understood as the complex tangent plane at w_0 to the d-dimensional complex manifold $\log \boldsymbol{Z}_f^{\mathrm{reg}}$.

where $\varphi_j, \psi_j \in \mathbb{R}^n$ and $w = u + iv$. Let $L_{\rm Re}$ and $L_{\rm Im}$ be the projections of T_0 on the real subpaces $\mathbb{R}^n_u$ and $\mathbb{R}^n_v$. As soon as $(u,v) \simeq u + iv$ belongs to T_0, so does $(-v, u)$ (since T_0 is also a complex tangent space to a d-dimensional submanifold of $\mathbb{C}^n_w$). Therefore the two subpaces $L_{\rm Re}$ and $L_{\rm Im}$ coincide as $\mathbb{R}$-linear subspace is one identifies the copies $\mathbb{R}^n_u$ and $\mathbb{R}^n_v$ of $\mathbb{R}^n$. Therefore

$$T_0 \subset L_{\rm Re} + iL_{\rm Im} = L_{\rm Re} + iL_{\rm Re}.$$

The vectors $\varphi_j + i\psi_j$ lie in the complex normal space to $\log \boldsymbol{Z}_f^{\rm reg}$ at 0, that is in $\gamma(z_0)$, and it remains to analyze whether they are real. Let us distinguish the two cases.

1. Suppose first that $2d \leq n$. We need to ensure that the tangent map Π_0 is not injective, which means that $L_{\rm Re}$ lies in a $(2d-1)$-$\mathbb{R}$-subspace such as

$$L'_{\rm Re} := \{u \in \mathbb{R}^n ; \, \langle \alpha_j, u\rangle = 0 \text{ for } j = 1, ..., n - 2d + 1\} \quad (\alpha_j \in (\mathbb{R}^n)^\star, j = 1, ..., n - 2d + 1)$$

   where the rank of the matrix α equals $n - 2d + 1$. Since $T_0 \subset L_{\rm Re} + iL_{\rm Re}$, one has

$$T_0 \subset \{w = u + iv \, ; \, \langle \alpha_j, u + iv\rangle = 0 \text{ for } j = 1, ..., n - 2d + 1\}.$$

   Since the complex subspace of the right-hand side is defined by linear equations with real coefficients, $\gamma(z_0)$ contains at least $n - 2d + 1$ independent real vectors.

2. Suppose now that $2d > n$. We need to ensure that the tangent map Π_0 is not surjective, which means that $L_{\rm Re}$ lies in a $\mathbb{R}$- hyperplane $\{u \in \mathbb{R}^n \, ; \, \langle \alpha, u\rangle = 0\}$ for some non-zero $\alpha \in (\mathbb{R}^n)^\star$. One concludes as before that the real vector α belongs to $\gamma(z_0)$.

This discussion concludes the proof of the theorem ■

0.0.1 Examples

The contour of the archimedean amœba $\mathscr{A}_f$ of an algebraic hypersurface in $\mathbb{T}^n$ contains the boundary of this amœba, that is is definitively non empty. In fact, the Gauss logarithmic map reveals to be in this case a convenient tool to draw the amœba[15].

The situation is indeed totally different for reduced algebraic complete intersections as soon as $n > 3$.

Let us focus on the case of one dimensional reduced algebraic intersections in $\mathbb{T}^n$, that is algebraic curves defined as a complete intersection in $\mathbb{T}^n$. The simplest example is that of complex lines as

$$\boldsymbol{Z}_f = \{z \in \mathbb{T}^n \, ; \, z_j = a_j z_1 + b_j \text{ for } j = 2, ..., n\} \quad (a_j, b_j \in \mathbb{T}). \tag{0.0.10}$$

One has in this case

$$\mathrm{Log}_{|\boldsymbol{Z}_f} = \big(\log|z_1|, \log|a_2 z_1 + b_1|, \cdots, \log|a_n z_1 + b_n|\big)_{|\boldsymbol{Z}_f}.$$

[15]We will propose in section ?? when $n = 2$ an alternative tool, namely the numerical approximation of the Ronkin function R_f.

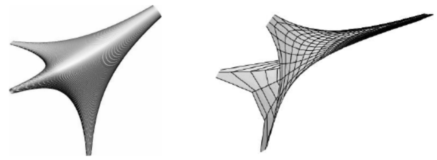

Figure 2

Its jacobian matrix, when expressed in the complex conjugate coordinates $z_1, \bar{z}_1$ (z_1 being a complex parameter on $\boldsymbol{Z}_f$) equals

$$\frac{D\,\mathrm{Log}_{|\boldsymbol{Z}_f}}{D\,(z_1,\bar{z}_1)} = \frac{1}{2}\begin{pmatrix} 1/z_1 & 1/\bar{z}_1 \\ 1/(z_2+w_2) & 1/(\bar{z}_2+\bar{w}_2) \\ \vdots & \vdots \\ 1/(z_n+w_n) & 1/(\bar{z}_n+\bar{w}_n) \end{pmatrix} \qquad (w_j = a_j/b_j = u_j + iv_j) \tag{0.0.11}$$

A point $(z_1, a_2z_1+b_2, ..., a_nz_1+b_n) \in \boldsymbol{Z}_f$ is critical for $\mathrm{Log}_{|\boldsymbol{Z}_f}$ if and only if the rank of the matrix (0.0.11) is strictly less than 2. If $z_1 = x+iy$, this is equivalent to say that

$$\begin{aligned} xv_j - yu_j &= 0 \quad (2 \le j \le n) \\ (xv_k - yu_k) - (xv_\ell - yu_\ell) &= u_kv_\ell - u_\ell v_k \quad (2 \le k < \ell \le n). \end{aligned}$$

Such a system is consistent if and only if $u_kv_\ell - u_\ell v_k = 0$ for any $2 \le k < \ell \le n$, that is

$$\frac{a_kb_\ell}{a_\ell b_k} \in \mathbb{R}^n \quad (2 \le k < \ell \le n). \tag{0.0.12}$$

Thus, one can state the following result [1].

Proposition 0.0.1 Let $n \ge 3$. The amœba $\mathscr{A}_f$ of the complex line of $\mathbb{T}^n$ defined as (0.0.10) is non empty if and only if the compatibility conditions (0.0.12) are fulfilled. If it is the case, the contour of the archimedean amœba $\mathscr{A}_f$ is the image of the real line

$$\{(x+iy, a_2(x+iy)+b_2, ..., a_n(x+iy)+b_n)\,;\, v_2x - u_2y = 0\} \subset \boldsymbol{Z}_f$$

by Log (where $a_j/b_j = u_j + iv_j$, $u_j, v_j \in \mathbb{R}$).

Examples 0.0.1

1. Let $n = 3$. The compatibility conditions (0.0.12) are not fulfilled when

$$\boldsymbol{Z}_f = \{z \in \mathbb{T}^3\,;\, z_2 = z_1 + 1,\ z_3 = z_1 + 1 + i\}.$$

   The contour is thus empty. The map Log realizes a (real) diffeomorphism between the real surface $(\boldsymbol{Z}_f)_{\mathbb{R}}$ and the amœba $\mathscr{A}_f$, which is then said to be *non-degenerated*. See Figure 2, left.

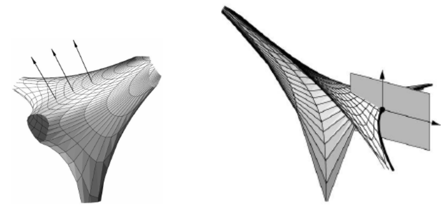

Figure 3

2. Let still $n = 3$. The compatibility conditions (0.0.12) are fulfilled when

$$\boldsymbol{Z}_f = \{z \in \mathbb{T}^3\,;\, z_2 = z_1 + 1,\ z_3 = z_1 + 1\}$$

since here $a_2 b_3/(a_3 b_2) = 2 \in \mathbb{R}$. The amœba is a surface with boundary in $\mathbb{R}^3$ and each of its interior points has two pre-images *via* Log on $\boldsymbol{Z}_f$: the images in $\mathbb{R}^3$ *via* Log of the conjugate points $(z_1, z_1 + 1, z_1 + 2)$ and $(\bar{z}_1, \bar{z}_1 + 1, \bar{z}_1 + 2)$ (distinct when $z_1 \notin \mathbb{R}$) do coincide. The contour of $\mathscr{A}_f$ is its topological boundary, which is realized as the image of the real line

$$\{(x + iy, (x + iy) + 1, (x + iy) + 2) \in \boldsymbol{Z}_f\,;\, y = 0\} \subset \boldsymbol{Z}_f$$

by Log. The amœba is realized from the non-degenerated one by "*collapsing*". Note that at any point z_0 such that $\mathrm{Log}(z_0)$ is on the contour, that is here the topological boundary, of the amœba $\mathscr{A}_f$, the image $\gamma(z_0)$, that is the normal complex space to $\log \boldsymbol{Z}_f$, contains a real plane. See Figure 3, right.

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