A Diophantine Demonstration of the non-monogeneity for Triquadratic Number Fileds

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Thanks

Thanks!

Before making my presentation, allow me to thank the organizers of this online meeting, and more particularly Professor Tony EZOME from the University of Libreville (GABON), for inviting me to give a talk , on the theme of monogenity, initiative that I find wonderful.

I therefore send them all my congratulations, my thanks and my encouragements.

Abstract

Let be a triquadratic number field
$K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$, with $(dm, dn, d'm'n'l)$ in all possible three occurring cases coverered by:

$$(dm, dn, d'm'n'l) \equiv (1, 1, 1), (1, 1, 2 \text{ or } 3) \text{ and } (1, 2, 3)\textbf{(mod4)}.$$

Those cases are respectively called **case 1**, **case 2** and **case 3**.

This work comes from on two articles written by Kouassi Vincent Kouakou and myself (cf [13])et (cf [14]). First, we characterize the fact that in K_3:
$\theta = \omega_1 + \omega_2\sqrt{dm} + \omega_3\sqrt{dn} + \omega_4\sqrt{mn} + \omega_5\sqrt{d'm'n'l} + \omega_6\sqrt{\frac{dm}{d'm'}n'l} + \omega_7\sqrt{\frac{dn}{d'n'}m'l} + \omega_8\sqrt{\frac{mn}{m'n'}d'l}$, where $\omega_i \in \mathbb{Q}$ is an algebraic integer of K_3 by showing the existence of $a_i \in \mathbb{Z}$ such that $\omega_i = \pm\frac{1}{8}a_i$ in the first case and $\omega_i = \pm\frac{1}{4}a_i$ in the two others. (The signs $\pm$ are easy to determine and depend on K_3). Those $a_i \in \mathbb{Z}$ must check a system of 8 linear congruencies which are **(mod2)**, **(mod4)** or **(mod8)**. After this characterization, we can put the monogenity's problem of K_3, i.e. can we find an algebraic integer $\theta \in K_3$ such that

$$discr(\theta) = D_{K_3/\mathbb{Q}} \tag{1}$$

If the equation (1) is solvable, then the set $\{1, \theta, ..., \theta^7\}$ is a basis of the integral ring $\mathbb{Z}_{K_3}$ of K_3. In this presentation, we'll resolve (1) only in the case $(dm, dn, d'm'n'l) \equiv (1, 1, 1)$**(mod4)**. We'll set by purly modular and diophantine calculations that (1) does not have solution.

For the two remaining cases, please see the comments at the end, where the essential points are given. **Finally, there will be only one triquadratic field which will be monogenic, and which will also belong to case 3. It is the cyclotomic field** $\mathbb{Q}(\zeta_{\mathbf{24}}) = \mathbb{Q}(\sqrt{-\mathbf{3}}, \sqrt{\mathbf{2}}, \sqrt{-\mathbf{1}})$

Introduction

Historically, I have been interested in a Diophantine demonstration of the monogeneity of triquadratic fields from 1991. Indeed during my thesis in Besançon, I had met Professeur Danielle Chatelain, who had given me a copy of her work, in which she had built an integer base for these fields. But it was late around 2008 that I find the time to work on this theme. And that's why I use the term Chatelain's bases for such fields.

Introduction

Let us put $K_n = \mathbb{Q}(\sqrt{A_{2^0}}, \sqrt{A_{2^1}}, ..., \sqrt{A_{2^{n-3}}}, \mu\sqrt{A_{2^{n-2}}}, \mu'\sqrt{A_{2^{n-1}}})$, for a general n-quadratic number field, where $(\mu, \mu') \in \{(1,1), (1, \sqrt{\pm 2}), (1, \sqrt{-1}), (\sqrt{\pm 2}, \sqrt{-1})\}$, and the $A_{2^k} \equiv 1 (\textbf{mod4})$: $0 \leq k \leq n-1$, are taken square free into $\mathbb{Z}^* \setminus \{1\}$, excepted in certain cases for which the quantities $A_{2^{n-2}}$ and $A_{2^{n-1}}$ can be equal to 1. For such number fields K_n, D. Chatelain (cf [1]), gave a method for the construction of an integral $\mathbb{Z}-$basis $\mathfrak{B}_{K_n}$ for its integral ring $\mathbb{Z}_{K_n}$, but also for calculations of $\mathfrak{D}_{K_n/\mathbb{Q}}$, the discriminant of the number field K_n relatively to $\mathbb{Q}$. We'll apply the result to the field $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$, $(dm, dn, d'm'n'l)$ in order to obtain a basis of integers of the integral ring $\mathbb{Z}_{K_3}$ in the case $(dm, dn, d'm'n'l) \equiv (1,1,1)(\textbf{mod4})$ (cf [1]), see the comments for the remainng two cases

Let us remark that Motoda and al. (cf [6]) and Gábor Nyul (cf [9]) gave also bases of triquadratic number fields, but they didn't explain the way they obtained them.
In particular Motoda and al. have solved the problem of the monogeneity of these fields, however with non-Diophantine and often relatively complicated arguments. Thus the case 1 which interests us, is treated from the theory of ramification quite simply in a very short theorem . But the same is not true for the other 2 cases . So our work is fully justified, especially for the other cases even if it is the cases 1 that I treat here.

Note that the only possible classifications for congruencies modulo 4, concerning the triplet (cf. [1] pp. 10-11 et [4] pp. 121) $(dm, dn, d'm'n'l)$ of the field $K_3 = \mathbb{Q}(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l})$ are: $(dm, dn, d'm'n'l) \equiv (1,1,1), (1,1,2 \text{ or } 3)$ and $(1,2,3)$**(mod4)** .

Let us recall, that we use, which we agreed once and for all (cf [5]), the following notations and conventions.

Notations

(i) Let's put $gcd(a,b) = (a,b), \forall a, b \in \mathbb{Z}$. When we write a triquadradic number field $K_3 = \mathbb{Q}(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l})$, that implies that d, m, n are square free rational integers, such that: $dm, dn, mn, d'm'n'l$ are $\neq 0$ and 1, with $(d,m) = (d,n) = (m,n) = (dmn, l) = 1$ and $d'|d, m'|m, n'|n$, and satisfying the conditions below.

$$(dm, dn, d'm'n'l) \equiv (1,1,1), (1,1,2 \text{ or } 3) \text{ and } (1,2,3)(\textbf{mod4}).$$

(ii) $s(a)$ is the sign of $a \in \mathbb{Z}^\star$.
(iii) For $a \equiv 1(\textbf{mod2})$, we put $\lambda_a \in \{-1, 1\}$, such that $a \equiv \lambda_a(\textbf{mod4})$.
$(iv) s_1 = \lambda_d s(d) = \pm 1, s_2 = \lambda_{d'm'} s(d'm') = \pm 1, s_3 = \lambda_{d'n'} s(d'n') = \pm 1$ and $s_4 = \lambda_{dm'n'} s(dm'n') = \pm 1$.

$(v)\ \gamma = \begin{cases} \lambda_{d'm'n'l} = -1, \text{ if } d'm'n'l \equiv -1 \textbf{ (mod4)} \\ \lambda_{d'm'n'\frac{l}{2}} = \pm 1, \text{ if } d'm'n'l \equiv 2 \textbf{ (mod4)}, l \text{ even} \end{cases},$

$(vi)\ \delta = \lambda_{d\frac{n}{2}}$ when n is even.

(vii) The Galois group $Gal(K_3/\mathbb{Q})$ on an α-basis of Chatelain $\{\alpha_i : 0 \leq \ i \ \leq 7\}$ of K_3 (cf [1], [5]) obtained via the α -matrix of Galois whose general term is:

$$a_{ji} = \frac{\sigma_i\,(\alpha_j)}{\alpha_j} = \pm 1, 0 \leq i, j \leq 7$$

and which is as follows: The Galois group acts on K_3 (which is a $\mathbb{Q}$-space vector set generated by 1 and the square roots just below) like this, and will work on the same way, hereafter, on the Chatelain's $\beta-$basis of K_3:

$$\begin{array}{cccccccc} \sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 \end{array} \qquad \alpha_i$$

$$M_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix} \qquad \begin{array}{r} 1 \\ \sqrt{dm} \\ \sqrt{dn} \\ \sqrt{mn} \\ \sqrt{d'm'n'l} \\ \sqrt{\frac{d}{d'}\frac{m}{m'}n'l} \\ \sqrt{\frac{d}{d'}\frac{n}{n'}m'l} \\ \sqrt{\frac{m}{m'}\frac{n}{n'}d'l} \end{array}$$

Lemma

For **Case 1**: $(dm, dn, d'm'n'l) \equiv (1,1,1)$ **(mod4)**
(i) A Chatelain's written form of $K_3 = \mathbb{Q}(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l})$ is $K_3 = \mathbb{Q}(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l})$.

(ii) The corresponding Chatelain's $\beta-$basis of K_3 is given by:

$$\beta = \left\{1, \sqrt{dm}, \sqrt{dn}, s_1\sqrt{mn}, \sqrt{d'm'n'l}, s_2\sqrt{\frac{d}{d'}\frac{m}{m'}n'l},\ s_3\sqrt{\frac{d}{d'}\frac{n}{n'}m'l}, s_4\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}\right\}$$

(iii) The corresponding Chatelain's $\mathbb{Z}-$base $\mathfrak{B}_{K_3}$ of $\mathbb{Z}_{K_3}$ is the set of the following elements:

$$\varepsilon_0 = \frac{1}{8}\left(1 + \sqrt{dm} + \sqrt{dn} + s_1\sqrt{mn} + \sqrt{d'm'n'l} + s_2\sqrt{\frac{d}{d'}\frac{m}{m'}n'l} + s_3\sqrt{\frac{d}{d'}\frac{n}{n'}m'l} + s_4\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}\right),$$

and its conjugates obtained from the $Gal(K_3/\mathbb{Q})$, (**1**). We make a $\mathbb{Z}$-transformation on the basis and have the following new one.

For **Case 1**:

$$\varepsilon_0' = \frac{1}{8}\left(1 + \sqrt{dm} + \sqrt{dn} + s_1\sqrt{mn} + \sqrt{d'm'n'l} + s_2\sqrt{\tfrac{dm}{d'm'}n'l} + s_3\sqrt{\tfrac{dn}{d'n'}m'l} + s_4\sqrt{\tfrac{mn}{m'n'}d'l}\right),$$

$$\varepsilon_1' = \frac{1}{8}\left(-2\sqrt{dm} - 2s_1\sqrt{mn} - 2s_2\sqrt{\tfrac{dm}{d'm'}n'l} - 2s_4\sqrt{\tfrac{mn}{m'n'}d'l}\right),$$

$$\varepsilon_2' = \frac{1}{8}\left(-2\sqrt{dn} + 2s_1\sqrt{mn} - 2s_3\sqrt{\tfrac{dn}{d'n'}m'l} + 2s_4\sqrt{\tfrac{mn}{m'n'}d'l}\right),$$

$$\varepsilon_3' = \frac{1}{8}\left(-4s_1\sqrt{mn} - 4s_4\sqrt{\tfrac{mn}{m'n'}d'l}\right),$$

$$\varepsilon_4' = \frac{1}{8}\left(-2\sqrt{d'm'n'l} - 2s_2\sqrt{\tfrac{dm}{d'm'}n'l} + 2s_3\sqrt{\tfrac{dn}{d'n'}m'l} + 2s_4\sqrt{\tfrac{mn}{m'n'}d'l}\right),$$

Lemma

$$\varepsilon'_5 = \frac{1}{8}\left(4s_2\sqrt{\tfrac{dm}{d'm'}n'l} - 4s_4\sqrt{\tfrac{mn}{m'n'}d'l}\right)$$
$$\varepsilon'_6 = \frac{1}{8}\left(-4s_3\sqrt{\tfrac{dn}{d'n'}m'l} + 4s_4\sqrt{\tfrac{mn}{m'n'}d'l}\right)$$
$$\varepsilon'_7 = \frac{1}{8}\left(-8s_4\sqrt{\tfrac{mn}{m'n'}d'l}\right).$$

The goal of this transformation is to make easy the calculation of the integers of K_3

Characterization of integers of $K_3 = \mathbb{Q}(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l})$

Let us take any element
$\theta = \omega_0 + \omega_1\sqrt{dm} + \omega_2\sqrt{dn} + \omega_3\sqrt{mn} + \omega_4\sqrt{d'm'n'l} + \omega_5\sqrt{\frac{dm}{d'm'}n'l} + \omega_6\sqrt{\frac{dn}{d'n'}m'l} + \omega_7\sqrt{\frac{mn}{m'n'}d'l}$ in K_3, where $\omega_i \in \mathbb{Q}$.
The problem which is set, is to know if θ is an integer of $K_3 = \mathbb{Q}(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l})$ or not.
The first thing to do is to distinguish among the 3 generic cases, **the case 1**, we are dealing with. To do this, let's take a look at the 7 quantities $dm, dn, mn, d'm'n'l, \frac{dm}{d'm'}n'l, \frac{dn}{d'n'}m'l, \frac{mn}{m'n'}d'l$. One remarks that:

1. Our case occurs if and only if more than 4 of the elements: $dm, dn, mn, d'm'n'l, \frac{dm}{d'm'}n'l, \frac{dn}{d'n'}m'l, \frac{mn}{m'n'}d'l$ are $\equiv 1(\mathbf{mod 4})$, exactly all terms are $\equiv 1(\mathbf{mod 4})$). Then let us remark that we write θ in the following theorem on the corresponding Chatelain's β-basis. We obtain:
2. In our case, $\theta = \omega_0 + \omega_1\sqrt{dm} + \omega_2\sqrt{dn} + \omega_3 s_1 \left(s_1\sqrt{mn}\right) + \omega_4\sqrt{d'm'n'l} + \omega_5 s_2 \left(s_2\sqrt{\frac{dm}{d'm'}n'l}\right) + \omega_6 s_3 \left(s_3\sqrt{\frac{dn}{d'n'}m'l}\right) + \omega_7 s_4 \left(s_4\sqrt{\frac{mn}{m'n'}d'l}\right), \omega_i \in \mathbb{Q}$.

From that, because we are able to write
$K_3 = \mathbb{Q}(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l})$ in a Chatelain's written form, it is clear that the following quantities are well known, once θ is given:
$s_1 = \lambda_d s(d) = \pm 1,$
$s_2 = \lambda_{d'm'} s(d'm') = \pm 1, s_3 = \lambda_{d'n'} s(d'n') = \pm 1,$
$s_4 = \lambda_{dm'n'} s(dm'n') = \pm 1,$

$$\gamma = \begin{cases} \lambda_{d'm'n'l} = -1, \ \textit{if} \ d'm'n'l \equiv -1 \ \textbf{(mod4)} \\ \lambda_{d'm'n'\frac{l}{2}} = \pm 1, \ \textit{if} \ d'm'n'l \equiv 2 \ \textbf{(mod4)} \end{cases},$$

and $\delta = \lambda_{d\frac{n}{2}}$ when n is even.

From all that we get the main theorem which caraterize the algebraic integers of K_3, when the case 1 is verified. For the two remaining cases see the comments.

Theorem

Let $K_3 = \mathbb{Q}(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l})$ a triquadratic number field checking **case 1** and satisfying Notations 1.1 (i), where $s_1, s_2, s_3, s_4, \gamma$ and δ are the signs coming from its Chatelain's written form. The following propositions are equivalent.
(i) $\theta = \omega_0 + \omega_1\sqrt{dm} + \omega_2\sqrt{dn} + \omega_3\sqrt{mn} + \omega_4\sqrt{d'm'n'l} + \omega_5\sqrt{\frac{dm}{d'm'}n'l} + \omega_6\sqrt{\frac{dn}{d'n'}m'l} + \omega_7\sqrt{\frac{mn}{m'n'}d'l}$, $\omega_i \in \mathbb{Q}$, is an integer of K_3 (i.e θ belongs to the ring of integral $\mathbb{Z}_{K_3}$ of K_3).

Theorem

(ii) $(cf.Lemma 1.1)$: There exists $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \in \mathbb{Z}^8$, such that:

$A_1)$ $\theta = \frac{1}{8}\left(a_0 + a_1\sqrt{dm} + a_2\sqrt{dn} + s_1 a_3\sqrt{mn} + a_4\sqrt{d'm'n'l} + s_2 a_5\sqrt{\frac{dm}{d'm'}n'l} + s_3 a_6\sqrt{\frac{dn}{d'n'}m'l} + s_4 a_7\sqrt{\frac{mn}{m'n'}d'l}\right)$, and

satisfying

Theorem

$$(A_2) \text{ And } : \begin{cases} a_0 \in \mathbb{Z} \\ a_0 - a_1 \equiv 0 \pmod{2} \\ a_2 - a_0 \equiv 0 \pmod{2} \\ a_0 + a_1 - a_2 - a_3 \equiv 0 (\textbf{mod4}) \\ a_0 - a_4 \equiv 0 \pmod{2} \\ a_0 - a_1 - a_4 + a_5 \equiv 0 (\textbf{mod4}) \\ a_0 + a_2 - a_4 - a_6 \equiv 0 (\textbf{mod4}) \\ a_0 + a_1 + a_2 + a_3 - a_4 - a_5 - a_6 - a_7 \equiv 0 (\textbf{mod8}). \end{cases}$$

(Note that $\omega_i = \dfrac{a_i}{8}$)

Proof

$(i) \Longrightarrow (ii)$

Let's suppose that θ is an integer of K_3. So:
$\theta = \omega_0 + \omega_1\sqrt{dm} + \omega_2\sqrt{dn} + \omega_3\sqrt{mn} + \omega_4\sqrt{d'm'n'l} + \omega_5\sqrt{\frac{dm}{d'm'}n'l} + \omega_6\sqrt{\frac{dn}{d'n'}m'l} + \omega_7\sqrt{\frac{mn}{m'n'}d'l} \in \mathbb{Z}_{K_3}$ with $\omega_i \in \mathbb{Q}$.
And following Remarks 2.2), we can rewrite θ on the Chatelain's β-basis of K_3 as following:

Proof

$$\theta = \omega_0 + \omega_1\sqrt{dm} + \omega_2\sqrt{dn} + \omega_3 s_1\left(s_1\sqrt{mn}\right) + \omega_4\sqrt{d'm'n'l} + \omega_5 s_2\left(s_2\sqrt{\frac{dm}{d'm'}n'l}\right) + \omega_6 s_3\left(s_3\sqrt{\frac{dn}{d'n'}m'l}\right) + \omega_7 s_4\left(s_4\sqrt{\frac{mn}{m'n'}d'l}\right).$$

We can also write the integer θ on the basis $\mathfrak{B}'_{K_3}$ of $\mathbb{Z}_{K_3}$ (cf. Lemma 1.4 1.). So there exists $(l_i)_{1\leq i\leq 7} \subset \mathbb{Z}$, such that:

$$\theta = l_0\varepsilon_0 + l_1\varepsilon'_1 + l_2\varepsilon'_2 + l_3\varepsilon'_3 + l_4\varepsilon'_4 + l_5\varepsilon'_5 + l_6\varepsilon'_6 + l_7\varepsilon'_7.$$

Proof

Let's develop and factorize this last expression on Chatelain's β-basis of K_3, then we get:

$$\theta = \frac{l_0}{8} + \frac{1}{8}\left(l_0 - 2l_1\right)\sqrt{dm} + \frac{1}{8}\left(l_0 - 2l_2\right)\sqrt{dn} +$$
$$\frac{1}{8}\left(l_0 - 2l_1 + 2l_2 - 4l_3\right)\left(s_1\sqrt{mn}\right)$$
$$+\frac{1}{8}\left(l_0 - 2l_4\right)\sqrt{d'm'n'l} + \frac{1}{8}\left(l_0 - 2l_1 - 2l_4 + 4l_5\right)\left(s_2\sqrt{\frac{dm}{d'm'}n'l}\right)$$
$$+\frac{1}{8}\left(l_0 - 2l_2 + 2l_4 - 4l_6\right)\left(s_3\sqrt{\frac{dn}{d'n'}m'l}\right)$$
$$+\frac{1}{8}\left(l_0 - 2l_1 + 2l_2 - 4l_3 + 2l_4 - 4l_5 + 4l_6 - 8l_7\right)\left(s_4\sqrt{\frac{mn}{m'n'}d'l}\right).$$

Proof

Let's put $(a_i)_{0\leq i\leq 7} \subset \mathbb{Z}$, such that:

$$\begin{cases} a_0 &= l_0 \\ a_1 &= l_0 - 2l_1 \\ a_2 &= l_0 - 2l_2 \\ a_3 &= l_0 - 2l_1 + 2l_2 - 4l_3 \\ a_4 &= l_0 - 2l_4 \\ a_5 &= l_0 - 2l_1 - 2l_4 + 4l_5 \\ a_6 &= l_0 - 2l_2 + 2l_4 - 4l_6 \\ a_7 &= l_0 - 2l_1 + 2l_2 - 4l_3 + 2l_4 - 4l_5 + 4l_6 - 8l_7. \end{cases}$$

Replace by these $(a_i)_{0\leq i\leq 7}$ the values $(l_i)_{0\leq i\leq 7}$ in the last expression of θ, let's identify these two expressions of θ, we get the point A_1) of (A).

Proof

Moreover, to prove the second point A_2) of (A), let's solve $(l_i)_{0\leq i\leq 7}$ from the $(a_i)_{0\leq i\leq 7}$, we get:

$$\begin{cases} l_0 &= a_0 \in \mathbb{Z} \\ l_1 &= \dfrac{a_0 - a_1}{2} \in \mathbb{Z} \\ l_2 &= \dfrac{a_0 - a_2}{2} \in \mathbb{Z} \\ l_3 &= \dfrac{a_0 + a_1 - a_2 - a_3}{4} \in \mathbb{Z} \\ l_4 &= \dfrac{a_0 - a_4}{2} \in \mathbb{Z} \\ l_5 &= \dfrac{a_0 - a_1 - a_4 + a_5}{4} \in \mathbb{Z} \\ l_6 &= \dfrac{a_0 + a_2 - a_4 - a_6}{4} \in \mathbb{Z} \\ l_7 &= \dfrac{a_0 + a_1 + a_2 + a_3 - a_4 - a_5 - a_6 - a_7}{8} \in \mathbb{Z}. \end{cases}$$

Proof

That is the requested system of congruences on the $(a_i)_{1\leq i\leq 8}$, of point A_2) .

$(ii) \Longrightarrow (i)$
Let's suppose that there exists $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \in \mathbb{Z}^8$, such that (ii) is realized. We have to show that θ is an integer of K_3. From the congruencies system, let's put the $(l_i)_{0\leq i\leq 0}$ in function of the $(a_i)_{0\leq i\leq 7}$, as done above.
Clearly, the $(l_i)_{0\leq i\leq 7} \subset \mathbb{Z}$. Now let's develop this following expression which is an integer of K_3:

$$l_0\varepsilon'_0 + l_1\varepsilon'_1 + l_2\varepsilon'_2 + l_3\varepsilon'_3 + l_4\varepsilon'_4 + l_5\varepsilon'_5 + l_6\varepsilon'_6 + l_7\varepsilon'_7.$$

Proof

We find that:

$$l_0\varepsilon'_0 + l_1\varepsilon'_1 + l_2\varepsilon'_2 + l_3\varepsilon'_3 + l_4\varepsilon'_4 + l_5\varepsilon'_5 + l_6\varepsilon'_6 + l_7\varepsilon'_7$$
$$= \frac{1}{8}(a_0 + a_1\sqrt{dm} + a_2\sqrt{dn} + s_1 a_3\sqrt{mn} + a_4\sqrt{d'm'n'l} + s_2 a_5\sqrt{\tfrac{dm}{d'm'}n'l}$$
$$+ s_3 a_6\sqrt{\tfrac{dn}{d'n'}m'l} + s_4 a_7\sqrt{\tfrac{mn}{m'n'}d'l})$$
$$= \theta.$$

So θ belong to $\mathbb{Z}_{K_3}$

Proof

In consequence, as announced in the point (*ii*) of **Case 1** .
$\theta = \omega_0 + \omega_1\sqrt{dm} + \omega_2\sqrt{dn} + \omega_3\sqrt{mn} + \omega_4\sqrt{d'm'n'l} + \omega_5\sqrt{\frac{dm}{d'm'}n'l} + \omega_6\sqrt{\frac{dn}{d'n'}m'l} + \omega_7\sqrt{\frac{mn}{m'n'}d'l}, \omega_i \in \mathbb{Q}$, is an integer of K_3.
(Exactly with $(\omega_0, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7) \in \mathbb{Q}^8$, defined by the equalities:
$(\omega_0, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7) = \left(\frac{1}{8}a_0, \frac{1}{8}a_1, \frac{1}{8}a_2, \frac{1}{8}s_1a_3, \frac{1}{8}a_4, \frac{1}{8}s_2a_5, \frac{1}{8}s_3a_6, \frac{1}{8}s_4a_7\right)$.

The problem of monogeneity

Let $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ be a triquadratic number field of odd discriminant, i.e. such that $(dm, dn, d'm'n'l) \equiv (1,1,1)$ **(mod4)** this means that **case 1** occurs. Then the discriminant of K_3 on $\mathbb{Q}$ is: $D_{K_3/\mathbb{Q}} = (dmnl)^4$ cf. [1]. We solve the problem of the existence or not of a power basis of the type $\left\{1, \theta, ..., \theta^7\right\}$, for the ring of integers $\mathbb{Z}_{K_3}$ which as we know is a $\mathbb{Z}$-module free of rank 8.

The classical method consists in solving an equivalent problem by solving in unknown $\theta \in \mathbb{Z}_{K_3}$, the classical monogeneity equation below, where $\sigma_i \in Gal(K_3\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^3$:

Let $\mathfrak{B}_{K_3} = \{\varepsilon_0, \varepsilon_1, \varepsilon_2, ..., \varepsilon_7\}$ be the $\mathbb{Z}$-basis of Chatelain of $\mathbb{Z}_{K_3}$ (cf. [5]), then the unknown $\theta \in \mathbb{Z}_{K_3}$ is written $\theta = \sum_{i=0}^{7} l_i \varepsilon_i$, where $l_i \in \mathbb{Z}$, and:

$$discr(\theta) = \prod_{0 \leq i < j \leq 7} (\sigma_i(\theta) - \sigma_j(\theta))^2 = I(l_1, ..., l_7)^2 D_{K_3/\mathbb{Q}}, \tag{2}$$

where I is a homogeneous form of $\mathbb{Z}[X_1, ..., X_7]$ of degree 28, called index form attached to the basis $\mathfrak{B}_{\mathbb{K}_3}$. Then the resolution of the equation of monogeneity

$$discr(\theta) = \pm D_{K_3/\mathbb{Q}},$$

returns from a diophantine point of view to solve in unknowns $(l_1, ..., l_7) \in \mathbb{Z}^7$, the following equation of monogeneity:

$$I(l_1, ..., l_7) = \pm 1. \tag{3}$$

This kind of problem has been solved for fields of small degrees, especially for biquadratic fields, among others by [2], [6].
This work is about of the solution of the problem of monogeneity of the fields of 8 degree, with Galois group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$.
We find same results, proved(by others methods) between 2002 and 2006 in [9], [7], [10], [8], [11] given in special cases and general ones.
However, in our methodology, unlike the previous authors, we use different methods, namely purely diophantine to solve the equation **(3)**, using modular calculations in $\mathbb{Z}/4\mathbb{Z}$.
Note that our method is efficient for three cases, in particular for **the case 3** which is the most complicated.

The principle of the proof

In a first step, see **definition 1**, after having agreed on a canonical writing for K_3, we generally construct an integer basis for any triquadratic field $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ using the works of D. Chatelain cf [1] on n-quadratic fields, which we apply to degree 8 cf [5]. We transform this basis of Chatelain, in a basis better adapted to the problem of the monogeneity, by scaling said basis. This will allow us to write much more simply the equation of monogeneity **(3)**, which finally splits into a system of seven Pell-Fermat's equations (S_1) (note that for the practical resolution we will only use the first three of these equations **(10)**). In general, this type of Pell-Fermat system either does not admit solutions, or when it admits, we get a unique solution cf [16], [3].

Writing conventions for the fields $K_3 = \mathbb{Q}(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l})$

Let $\mathbb{Q}\left(\sqrt{dm}\right), \mathbb{Q}\left(\sqrt{dn}\right), \mathbb{Q}\left(\sqrt{d'm'n'l}\right)$ be three quadratic subfields of K_3, distinct in pairs such that: $(m, n) = 1$, $(dmn, l) = 1$, $d' = (d, d'm'n'l)$, $m' = (m, d'm'n'l)$ and $n' = (n, d'm'n'l)$.

Then, according to the definition of Chatelain cf [1], the seven quadratic subfields of $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ are:

$$k_1 = \mathbb{Q}\left(\sqrt{dm}\right), \qquad k_2 = \mathbb{Q}\left(\sqrt{dn}\right), \qquad k_3 = \mathbb{Q}\left(\sqrt{mn}\right),$$

$$k_4 = \mathbb{Q}\left(\sqrt{d'm'n'l}\right), \quad k_5 = \mathbb{Q}\left(\sqrt{\frac{dm}{d'm'}n'l}\right), \quad k_6 = \mathbb{Q}\left(\sqrt{\frac{dn}{d'n'}m'l}\right)$$

and $$k_7 = \mathbb{Q}\left(\sqrt{\frac{mn}{m'n'}d'l}\right)$$

We deduce the seven biquadratic subfields of K_3.

$$a)\ K_1 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}\right) = \mathbb{Q}\left(\sqrt{md}, \sqrt{mn}\right) = \mathbb{Q}\left(\sqrt{nd}, \sqrt{nm}\right),$$

$$\text{b)} \quad K_2 = \mathbb{Q}\left(\sqrt{(d'm')\frac{dm}{d'm'}}, \sqrt{(d'm')\, n'l}\right)$$

$$= \mathbb{Q}\left(\sqrt{\left(\frac{dm}{d'm'}\right) d'm'}, \sqrt{\left(\frac{dm}{d'm'}\right) n'l}\right)$$

$$= \mathbb{Q}\left(\sqrt{(n'l)\, d'm'}, \sqrt{(n'l)\frac{dm}{d'm'}}\right)$$

$$c)\quad K_3=\mathbb{Q}\left(\sqrt{\left(\frac{d}{d'}m'\right)d'\frac{m}{m'}},\sqrt{\left(\frac{d}{d'}m'\right)\frac{n}{n'}l}\right)$$

$$=\mathbb{Q}\left(\sqrt{\left(\frac{m}{m'}d'\right)\frac{d}{d'}m'},\sqrt{\left(\frac{m}{m'}d'\right)\frac{n}{n'}l}\right)$$

$$=\mathbb{Q}\left(\sqrt{\left(\frac{n}{n'}l\right)\frac{d}{d'}m},\sqrt{\left(\frac{n}{n'}l\right)\frac{m}{m'}d'}\right),$$

$$d)\quad K_4=\mathbb{Q}\left(\sqrt{(d'n')\frac{dn}{d'n'}},\sqrt{(d'n')\,m'l}\right)$$

$$=\mathbb{Q}\left(\sqrt{\left(\frac{dn}{d'n'}\right)d'n'},\sqrt{\left(\frac{dn}{d'n'}\right)m'l}\right)$$

$$=\mathbb{Q}\left(\sqrt{(m'l)\,d'n'},\sqrt{(m'l)\frac{dn}{d'n'}}\right)$$

$$e)\quad K_5 = \mathbb{Q}\left(\sqrt{\left(d'\frac{n}{n'}\right)\frac{d}{d'}n'}, \sqrt{\left(d'\frac{n}{n'}\right)\frac{m}{m'}l}\right)$$

$$= \mathbb{Q}\left(\sqrt{\left(\frac{d}{d'}n'\right)d'\frac{n}{n'}}, \sqrt{\left(\frac{d}{d'}n'\right)\frac{m}{m'}l}\right)$$

$$= \mathbb{Q}\left(\sqrt{\left(\frac{m}{m'}l\right)\frac{d}{d'}n'}, \sqrt{\left(\frac{m}{m'}l\right)d'\frac{n}{n'}}\right),$$

$$f)\quad K_6 = \mathbb{Q}\left(\sqrt{(m'n')\frac{mn}{m'n'}}, \sqrt{(m'n')\,d'l}\right)$$

$$= \mathbb{Q}\left(\sqrt{\left(\frac{mn}{m'n'}\right)m'n'}, \sqrt{\left(\frac{mn}{m'n'}\right)d'l}\right)$$

$$= \mathbb{Q}\left(\sqrt{\left(\frac{d}{d'}l\right)\frac{m}{m'}n'}, \sqrt{\left(\frac{d}{d'}l\right)m'\frac{n}{n'}}\right)$$

and

$$g) \quad K_7 = \mathbb{Q}\left(\sqrt{\left(\frac{m}{m'}n'\right)m'\frac{n}{n'}}, \sqrt{\left(\frac{m}{m'}n'\right)\frac{d}{d'}l}\right)$$
$$= \mathbb{Q}\left(\sqrt{\left(\frac{n}{n'}m'\right)\frac{m}{m'}n'}, \sqrt{\left(\frac{n}{n'}m'\right)\frac{d}{d'}l}\right)$$
$$= \mathbb{Q}\left(\sqrt{(d'l)\,m'n'}, \sqrt{(d'l)\,\frac{mn}{m'n'}}\right).$$

2) Each of these seven biquadratic subfields $K_i = \mathbb{Q}\left(\sqrt{d_i m_i}, \sqrt{d_i n_i}\right)$ can be written in its canonical form cf [2], which means that $d_i m_i \equiv d_i n_i$ **(mod4)**$, 0 < d_i$, possibly even, $n_i < m_i$ odd (and when $d_i m_i \equiv d_i n_i \equiv 1$**(mod4)**, we take $d_i < \inf\left(|m_i|, |n_i|\right)$.

Formulas

We have etablished the important formulas which are the keys, of the methods of solving.

$$\begin{array}{rccc} \gamma_a : & 2\mathbb{Z}+1 & \longrightarrow & \mathbb{Z} \\ & a & \longmapsto & \dfrac{a-\lambda_a}{4} \end{array}.$$

1) Let $a \equiv 1$ **(mod2)** , we write $\lambda_a \in \{-1;1\}$, such that $a \equiv \lambda_a$**(mod4)**, then

$$a = \lambda_a + 4\gamma_a.$$

2) $s(a)$ the sign of a $a \in \mathbb{Z}^{\star}$. 3) Let's note that $a \equiv 1 \pmod{2} \implies \lambda_a a \equiv 1 \pmod{4}$.
4) $\forall a,\ b \in 2\mathbb{Z}+1$ then $\lambda_{ab} = \lambda_a \lambda_b$. In particular $\lambda_{a^2} = 1$ et $\lambda_{a^2 b} = \lambda_b$.
5) $\forall a, b \in 2\mathbb{Z}+1$, $\lambda_{ab} = 1 \Leftrightarrow \lambda_a = \lambda_b$.

6) In particular the following equalities hold:

$$\lambda_{dm} = \lambda_{dn} = \lambda_{mn} = \lambda_{d'm'n'l} = \lambda_{\frac{dm}{d'm'}n'l} = \lambda_{\frac{d}{d'}m'\frac{n}{n'}l} = \lambda_{d'\frac{m}{m'}\frac{n}{n'}l} = 1,$$

and will allow useful factorizations via point 5).
7) Let a and b be odd, then:

$$\gamma_{ab} \equiv \lambda_a \gamma_b + \lambda_b \gamma_a \mathbf{(mod 4)}.$$

Moreover, $\forall c \in \mathbb{Z}$ we have:

$$2c(\lambda_a \pm \lambda_b) \equiv 0 \mathbf{(mod 4)}.$$

Chatelain's canonical writing of $K_3 = \mathbb{Q}(\sqrt{dm}, \sqrt{dn}\sqrt{d'm'n'l})$

Let take $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ with $(dm, dn, d'm'n'l) \equiv (1,1,1)\,\textbf{(mod4)}$, then we give a canonical writing of K_3 as follows:

(i) We first choose $K_2 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}\right)$ written in biquadratic canonical form such that $0 < d = \inf(d_i)$ and with maximal m among the eligible m_i choices and maximum $n < m$ among the remaining n_i.

(ii) Then choose $\mathbb{Q}\left(\sqrt{d'm'n'l}\right)$ among the four remaining quadratic fields such as:

$$d'm'n'l = \inf\left\{d'm'n'l, \frac{dm}{d'm'}n'l, \frac{d}{d'}m'\frac{n}{n'}l, d'\frac{m}{m'}\frac{n}{n'}l\right\}$$

and

$$s(d') = s(m') = s(n').$$

These conditions are always possible to be realized

Proposition

Let $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ *written in canonical form. Then*

$$s(d) = 1 \text{ and } s_1 = \lambda_d, s_2 = \lambda_{d'm'}, s_3 = \lambda_{d'n'}, s_4 = \lambda_{dm'n'}.$$

Monogeneity equations

On this new scaled basis $\mathfrak{B}'_{K_3}$ of $\mathbb{Z}_{K_3}$, let us take $\theta \in \mathbb{Z}_{\mathbb{K}_3}$, then there exist $l_0,\ l_1,\ l_2,\ l_3,\ l_4,\ l_5,\ l_6,\ l_7 \in \mathbb{Z}$ such that:

$$\theta = l_0\varepsilon'_0 + ... + l_7\varepsilon'_7 \ . \tag{4}$$

The equation of monogeneity **(3)** is written:

$$I\left(l_1, ..., l_7\right) = \pm 1. \tag{5}$$

For the actual calculation, we come back to the Chatelain $\beta-$basis of K_3:

$$\begin{aligned}\theta &= \frac{l_0}{8} + \frac{1}{8}(l_0 - 2l_1)\sqrt{dm} + \frac{1}{8}(l_0 - 2l_2)\sqrt{dn} \\ &+ \frac{1}{8}(l_0 - 2l_1 + 2l_2 - 4l_3)\, s_1\sqrt{mn} \\ &+ \frac{1}{8}(l_0 - 2l_4)\sqrt{d'm'n'l} + \frac{1}{8}(l_0 - 2l_1 - 2l_4 + 4l_5)\, s_2\sqrt{\frac{dm}{d'm'}n'l} \\ &+ \frac{1}{8}(l_0 - 2l_2 + 2l_4 - 4l_6)\, s_3\sqrt{\frac{dn}{d'n'}m'l} \\ &+ \frac{1}{8}(l_0 - 2l_1 + 2l_2 - 4l_3 + 2l_4 - 4l_5 + 4l_6 - 8l_7)\, s_4\sqrt{\frac{mn}{m'n'}d'l}.\end{aligned}$$

Let $a_i \in \mathbb{Z}$, $i = 0, ..., 7$, such that:

$$\begin{cases} a_0 = l_0 \\ a_1 = l_0 - 2l_1 \\ a_2 = l_0 - 2l_2 \\ a_3 = l_0 - 2l_1 + 2l_2 - 4l_3 \\ a_4 = l_0 - 2l_4 \\ a_5 = l_0 - 2l_1 - 2l_4 + 4l_5 \\ a_6 = l_0 - 2l_2 + 2l_4 - 4l_6 \\ a_7 = l_0 - 2l_1 + 2l_2 + 2l_4 - 4l_3 - 4l_5 + 4l_6 - 8l_7 \end{cases} . \qquad (6)$$

Note that conversely for these same a_i :

$$\begin{cases} l_0 = a_0 \\ l_1 = \dfrac{a_0 - a_1}{2} \\ l_2 = \dfrac{a_0 - a_2}{2} \\ l_3 = \dfrac{a_0 + a_1 - a_2 - a_3}{4} \\ l_4 = \dfrac{a_0 - a_4}{2} l_5 = \dfrac{a_0 - a_1 - a_4 + a_5}{4} \\ l_6 = \dfrac{a_0 + a_2 - a_4 - a_6}{4} \\ l_7 = \dfrac{a_0 + a_1 + a_2 + a_3 - a_4 - a_5 - a_6 - a_7}{8} \end{cases} \tag{7}$$

So that:

$$\begin{aligned}\theta &= \frac{1}{8}a_0 + \frac{1}{8}a_1\sqrt{dm} + \frac{1}{8}a_2\sqrt{dn} + \frac{1}{8}a_3 s_1\sqrt{mn} \\ &+ \frac{1}{8}a_4\sqrt{d'm'n'l} + \frac{1}{8}a_5 s_2\sqrt{\frac{dm}{d'm'}n'l} \\ &+ \frac{1}{8}a_6 s_3\sqrt{\frac{dn}{d'n'}m'l} + \frac{1}{8}a_7 s_4\sqrt{\frac{mn}{m'n'}d'l}.\end{aligned}$$

As a result:

$$I^2(l_1, ..., l_7) = I^2(a_1, ...a_7)$$

So that, solve this equation is equivalent to solve the following:

$$I(a_1, ..., a_7) = \pm 1 \tag{8}$$

Calculation of $\Delta(\theta) = discr(\theta)$

1) We calculate

$$\Delta(\theta) = discr\,(\theta) = \prod_{0 \leq i < j \leq 7} (\sigma_i\,(\theta) - \sigma_j\,(\theta))^2$$

in terms of $I\,(a_1, ..., a_7)$, the variable a_0 disappearing, cf **remark 1**. So we can take $a_0 = l_0 = 0$, in **(6)**, without affecting the generality of our resolution. So in the following we will have:

$$\theta = \frac{1}{8}a_1\sqrt{dm} + \frac{1}{8}a_2\sqrt{dn} + \frac{1}{8}a_3 s_1\sqrt{mn} + \frac{1}{8}a_4\sqrt{d'm'n'l} + \frac{1}{8}a_5 s_2\sqrt{\tfrac{dm}{d'm'}n'l} + \frac{1}{8}a_6 s_3\sqrt{\tfrac{dn}{d'n'}m'l} + \frac{1}{8}a_7 s_4\sqrt{\tfrac{mn}{m'n'}d'l}$$ with:

$$\begin{cases} a_1 = -2l_1;\ a_2 = -2l_2;\ a_3 = -2l_1 + 2l_2 - 4l_3;\ a_4 = -2l_4; \\ a_5 = -2l_1 - 2l_4 + 4l_5;\ a_6 = -2l_2 + 2l_4 - 4l_6; \\ a_7 = -2l_1 + 2l_2 + 2l_4 - 4l_3 - 4l_5 + 4l_6 - 8l_7. \end{cases}$$

If later we want to give the general solution θ including $a_0 = l_0$ we will use formulas $(7), (6)$ and (4).
2) For the calculations of $\Delta(\theta)$, let us make the following groupings and define by the same the following pairs of $\mathbb{Z}^2$: (A_1, C_1), (B_1, D_1), (E_1, F_1), (G_1, H_1), (I_1, J_1), (K_1, L_1), (M_1, N_1), as well as the numbers of K_3 : θ_1, θ_2, θ_3, θ_4, θ_5, θ_6 and θ_7 as follows:

• $\Delta_1 = (\sigma_0(\theta) - \sigma_2(\theta)) \times (\sigma_4(\theta) - \sigma_5(\theta)) \times (\sigma_1(\theta) - \sigma_3(\theta)) \times (\sigma_5(\theta) - \sigma_7(\theta))$

$$= (\frac{n}{n'})^2 \times$$

$$N_{k_1/\mathbb{Q}} \left(\frac{\frac{a_2^2 dn' + a_3^2 mn' - a_6^2 \frac{d}{d'} m' l - a_7^2 \frac{m}{m'} d' l}{8} + \frac{a_2 a_3 s_1 n' - a_6 a_7 s_3 s_4 l}{4} \sqrt{dm}}{2} \right)$$

$$= (\frac{n}{n'})^2 \times N_{k_1/\mathbb{Q}} \left(\frac{A_1 + C_1\sqrt{dm}}{2} \right) = (\frac{n}{n'})^2 \times N_{k_1/\mathbb{Q}}(\theta_1);$$

$$\bullet\ \Delta_2 = (\sigma_0(\theta) - \sigma_6(\theta)) \times (\sigma_2(\theta) - \sigma_4(\theta))(\sigma_1(\theta) - \sigma_7(\theta)) \times (\sigma_3(\theta) - \sigma_5(\theta))$$

$$= (n')^2 \times N_{k_1/\mathbb{Q}}\left(\frac{\frac{a_2^2 d\frac{n}{n'} + a_3^2 m\frac{n}{n'} - a_4^2 d'm'l - a_5^2\frac{dm}{d'm'}l}{8} + \frac{a_2a_3s_1\frac{n}{n'} - a_4a_5s_3l}{4}\sqrt{dm}}{2}\right)$$

$$= (n')^2 \times N_{k_1/\mathbb{Q}}\left(\frac{B_1 + D_1\sqrt{dm}}{2}\right) = (n')^2 \times N_{k_1/\mathbb{Q}}(\theta_2);$$

$$\Delta_3 = (\sigma_0(\theta) - \sigma_4(\theta)) \times (\sigma_2(\theta) - \sigma_6(\theta)) \times (\sigma_1(\theta) - \sigma_5(\theta)) \times (\sigma_3(\theta) - \sigma_7(\theta))$$

$$= (l)^2 \times N_{k_1/\mathbb{Q}}\left(\frac{\frac{a_4^2 d' m' n' + a_5^2 \frac{dm}{d'm'} n' - a_6^2 \frac{dn}{d'n'} m' - a_7^2 \frac{mn}{m'n'} d'}{8} + \frac{a_4 a_5 s_2 n' - a_6 a_7 s_3 s_4 \frac{n}{n'}}{4}\sqrt{dm}}{2}\right)$$

$$= (l)^2 \times N_{k_1/\mathbb{Q}}\left(\frac{E_1 + F_1\sqrt{dm}}{2}\right) = (l)^2 \times N_{k_1/\mathbb{Q}}(\theta_3);$$

- $\Delta_4 = (\sigma_0(\theta) - \sigma_1(\theta)) \times (\sigma_4(\theta) - \sigma_5(\theta)) \times (\sigma_2(\theta) - \sigma_3(\theta)) \times (\sigma_6(\theta) - \sigma_7(\theta))$

$= (\frac{m}{m'})^2 \times$

$$N_{k_2/\mathbb{Q}}\left(\frac{\frac{a_1^2 dm' + a_3^2 m' n - a_5^2 \frac{d}{d'} n' l - a_7^2 \frac{n}{n'} d' l}{8} + \frac{a_1 a_3 s_1 m' - a_5 a_7 s_2 s_4 l}{4}\sqrt{dn}}{2}\right)$$

$$= (\frac{m}{m'})^2 \times N_{k_2/\mathbb{Q}}\left(\frac{G_1 + H_1\sqrt{dn}}{2}\right) = (\frac{m}{m'})^2 \times N_{k_2/\mathbb{Q}}(\theta_4);$$

$$\bullet\ \Delta_5 = (\sigma_0(\theta) - \sigma_5(\theta)) \times (\sigma_1(\theta) - \sigma_4(\theta)) \times (\sigma_2(\theta) - \sigma_7(\theta)) \times (\sigma_3(\theta) - \sigma_6(\theta))$$

$$= (m')^2 \times N_{k_2/\mathbb{Q}}\left(\frac{\frac{a_1^2 d\frac{m}{m'} + a_3^2\frac{m}{m'}n - a_4^2 d'n'l - a_6^2\frac{dn}{d'n'}l}{8} + \frac{a_1a_3s_1\frac{m}{m'} - a_4a_6s_3l}{4}\sqrt{dn}}{2}\right)$$

$$= (m')^2 \times N_{k_2/\mathbb{Q}}\left(\frac{I_1 + J_1\sqrt{dn}}{2}\right) = (m')^2 \times N_{k_2/\mathbb{Q}}(\theta_5);$$

• $\Delta_6 = (\sigma_0(\theta) - \sigma_7(\theta)) \times (\sigma_1(\theta) - \sigma_6(\theta)) \times (\sigma_2(\theta) - \sigma_5(\theta)) \times (\sigma_3(\theta) - \sigma_4(\theta))$

$$= (d')^2 \times N_{k_3/\mathbb{Q}} \left(\frac{\frac{a_1^2 \frac{d}{d'} m + a_2^2 \frac{d}{d'} n - a_4^2 m' n' l - a_7^2 \frac{mn}{m'n'} l}{8} + \frac{a_1 a_2 \frac{d}{d'} - a_4 a_7 s_4 l}{4} \sqrt{mn}}{2} \right)$$

$$= (d')^2 \times N_{k_3/\mathbb{Q}} \left(\frac{K_1 + L_1 \sqrt{mn}}{2} \right) = (d')^2 \times N_{k_3/\mathbb{Q}}(\theta_6);$$

$$\bullet\ \Delta_7 = (\sigma_0(\theta) - \sigma_3(\theta)) \times (\sigma_4(\theta) - \sigma_7(\theta)) \times (\sigma_5(\theta) - \sigma_6(\theta)) \times (\sigma_1(\theta) - \sigma_2(\theta))$$

$$= (\frac{d}{d'})^2 \times$$

$$N_{k_3/\mathbb{Q}} \left(\frac{\frac{a_1^2 d' m + a_2^2 d' n - a_5^2 \frac{m}{m'} n' l - a_6^2 \frac{n}{n'} m' l}{8} + \frac{a_1 a_2 d' - a_5 a_6 s_2 s_3 l}{4} \sqrt{mn}}{2} \right);$$

$$= (\frac{d}{d'})^2 \times N_{k_3/\mathbb{Q}} \left(\frac{M_1 + N_1 \sqrt{mn}}{2} \right) = (\frac{d}{d'})^2 \times N_{k_3/\mathbb{Q}}(\theta_7);$$

It is clear that:

$$discr\,(\theta) = (\Delta_1 \times \Delta_2 \times \Delta_3 \times \Delta_4 \times \Delta_5 \times \Delta_6 \times \Delta_7)^2\,.$$

And that we have:

$$\prod_{0 \leq i < j \leq 7} (\sigma_i(\theta) - \sigma_j(\theta)) = (dmnl)^2 \times N_{k_1/\mathbb{Q}}(\theta_1) \times N_{k_1/\mathbb{Q}}(\theta_2) \times N_{k_1/\mathbb{Q}}(\theta_3) \times N_{k_2/\mathbb{Q}}(\theta_4) \times N_{k_2/\mathbb{Q}}(\theta_5) \times N_{k_3/\mathbb{Q}}(\theta_6) \times N_{k_3/\mathbb{Q}}(\theta_7).$$

Thus:

Proposition

The powers of $\theta \in \mathbb{Z}_{K_3}$ *form a basis of* $\mathbb{Z}_{K_3}$ *if and only if*

$$N_{k_1/\mathbb{Q}}(\theta_1) \times N_{k_1/\mathbb{Q}}(\theta_2) \times N_{k_1/\mathbb{Q}}(\theta_3) \times N_{k_2/\mathbb{Q}}(\theta_4) \times N_{k_2/\mathbb{Q}}(\theta_5) \times N_{k_3/\mathbb{Q}}(\theta_6) \times N_{k_3/\mathbb{Q}}(\theta_7) = \pm 1 \quad (9)$$

Equation equivalent to (8) : $I(a_1, ..., a_7) = \pm 1$.

And, we obtain the following system:

$$(S_1) : \begin{cases} N_{\mathbb{Q}(\sqrt{dm})/\mathbb{Q}}(\theta_1) = \epsilon_1 \\ N_{\mathbb{Q}(\sqrt{dm})/\mathbb{Q}}(\theta_2) = \epsilon_2 \\ N_{\mathbb{Q}(\sqrt{dm})/\mathbb{Q}}(\theta_3) = \epsilon_3 \\ N_{\mathbb{Q}(\sqrt{dn})/\mathbb{Q}}(\theta_4) = \epsilon_4 \\ N_{\mathbb{Q}(\sqrt{dn})/\mathbb{Q}}(\theta_5) = \epsilon_5 \\ N_{\mathbb{Q}(\sqrt{mn})/\mathbb{Q}}(\theta_6) = \epsilon_6 \\ N_{\mathbb{Q}(\sqrt{mn})/\mathbb{Q}}(\theta_7) = \epsilon_7 \end{cases},$$

where $\epsilon_k = \pm 1, k = 0, 1, ..., 7.$

System of monogeneity equations

Let us show that in the product $I(a_1, ..., a_7)$, each factor is in $\mathbb{Z}$ and consequently is necessarily equal to ± 1. This will give us the system (S_1) .

• The detailed calculation hereafter show that the numbers A_1, $C_1 ..., M_1$, N_1 are in $\mathbb{Z}$

As an example we write A_1 et C_1 :

$$A_1 = \lambda_{dn'}(l_1 l_4 - l_4^2) + \mathbf{2}(\lambda_{dn'}(l_1 l_3 - l_1 l_5 + l_1 l_6 - l_2 l_3 - l_2 l_4 + l_2 l_5 + l_4 l_5 + l_3^2 - l_5^2) + \gamma_{dn'} l_2^2 + \gamma_{mn'}(l_1^2 + l_2^2) + \gamma_{\frac{d}{d'}m'l}(-l_2^2 + l_4^2) + \gamma_{\frac{m}{m'}d'l}(-l_1^2 - l_2^2 - l_4^2)) + \mathbf{4}(\lambda_{\frac{m}{m'}d'l}(-l_1 l_7 - l_2 l_6 + l_2 l_7 + l_4 l_7 + l_5 l_6 - l_6^2) + \gamma_{\frac{d}{d'}m'l}(l_2 l_4) + \gamma_{\frac{m}{m'}d'l}(l_1 l_2 + l_1 l_4 - l_2 l_4)) + \mathbf{8}(\gamma_{mn'}(l_1 l_3 - l_2 l_3) + \gamma_{\frac{d}{d'}m'l}(-l_2 l_6 + l_4 l_6 - l_6^2) + \gamma_{\frac{m}{m'}d'l}(-l_1 l_7 + l_2 l_7 + l_4 l_7 + l_5 l_6)) + \mathbf{16}(-l_1 l_7 + l_2 l_7 + l_4 l_7 + l_5 l_6) + \mathbf{32}(-l_7^2 - l_5 l_7 + l_6 l_7)) \quad \text{and}$$

System of monogeneity equations

$$C_1 = \lambda_{dn'}(l_1 l_4 - l_4^2) + \mathbf{2}\lambda_{dn'}(-l_1 l_6 + l_2 l_3 - l_2 l_5 + l_4 l_5) + \mathbf{4}(\lambda_{\frac{m}{m'}d'l}(l_2 l_6 - l_2 l_7 + l_4 l_7 - l_5 l_6 + l_6^2) + \lambda_d \gamma_{n'}(l_1 l_2 - l_2^2) + \lambda_{\frac{d}{d'}m'}\gamma_l(-l_1 l_2 + l_1 l_4 + l_2^2 - l_4^2)) + \mathbf{8}(-\lambda_{\frac{m}{m'}d'l} l_6 l_7 + \lambda_d \gamma_{n'} l_2 l_3 + \lambda_{\frac{d}{d'}m'}\gamma_l(-l_1 l_6 - l_2 l_5 + l_4 l_5)) + \mathbf{16}(\lambda_{\frac{d}{d'}m'}\gamma_l(l_2 l_6 - l_2 l_7 + l_4 l_7 - l_5 l_6 + l_6^2)) - \mathbf{32}\lambda_{\frac{d}{d'}m'}\gamma_l l_6 l_7.$$

System of monogeneity equations

• Moreover, the components of the following couples: (A_1, C_1), (B_1, D_1), (E_1, F_1), (G_1, H_1), (I_1, J_1), (K_1, L_1), (M_1, N_1) have the same parity. Indeed:

$A_1 - C_1 \equiv \lambda_{dn'}(l_1 l_4 - l_4^2) - \lambda_{dn'}(l_1 l_4 - l_4^2) \equiv 0 \ (\textbf{ mod2 }) ;$
$B_1 - D_1 \equiv \lambda_{d'm'l}(-l_1 l_2 + l_2^2 - l_1 l_4 - l_4^2) - \lambda_{d'm'l}(l_1 l_2 - l_2^2 - l_1 l_4 - l_4^2) \equiv 0 \ (\textbf{ mod2 }) ;$

System of monogeneity equations

$$E_1 - F_1 \equiv \lambda_{d'm'n'}(l_1 l_2 - l_2^2) - \lambda_{d'm'n'}(-l_1 l_2 + l_2^2) \equiv 0 \pmod{2}$$
$$G_1 - H_1 \equiv \lambda_{dm'}(-l_2 l_4 - l_4^2) - \lambda_{dm'}(l_2 l_4 + l_4^2) \equiv 0 \pmod{2} ;$$
$$I_1 - J_1 \equiv \lambda_{d\frac{m}{m'}}(-l_1 l_2 + l_1^2 + l_2 l_4 - l_4^2) - \lambda_{d\frac{m}{m'}}(-l_1 l_2 + l_1^2 - l_2 l_4 + l_4^2) \equiv 0 \pmod{2} ;$$
$$K_1 - L_1 \equiv \lambda_{\frac{d}{d'}m}(l_1 l_2 + l_1 l_4 - l_2 l_4 - l_4^2) - \lambda_{\frac{d}{d'}}(l_1 l_2 - l_1 l_4 + l_2 l_4 + l_4^2) \equiv 0 \pmod{2} ;$$
$$M_1 - N_1 \equiv \lambda_{d'm}(-l_1 l_4 + l_2 l_4 - l_4^2) - \lambda_{d'}(l_1 l_4 - l_2 l_4 + l_4^2) \equiv 0 \pmod{2} .$$

Note that for our proof, we will use only the reduced modulo 4 computation of the values of the first three couples, whose expressions are:

We have the following relationships modulo 4 :

$$\begin{cases} A_1 \equiv \lambda_{dn'}(l_1 l_4 - l_4^2) + 2\lambda_{dn'}(l_1 l_3 - l_1 l_5 + l_1 l_6 - l_2 l_3 - l_2 l_4 + l_2 l_5 + l_4 l_5 + l_3^2 - \\ \qquad l_5^2) + 2\gamma_{dn'} l_2^2 + 2\gamma_{mn'}(l_1^2 + l_2^2) + 2\gamma_{\frac{d}{d'}m'l}(-l_2^2 + l_4^2) + \\ \qquad 2\gamma_{\frac{m}{m'}d'l}(-l_1^2 - l_2^2 - l_4^2) \ \textbf{(mod4)} \\ B_1 \equiv \lambda_{d\frac{n}{n'}}(-l_1 l_2 + l_2^2 - l_1 l_4 - l_4^2) + 2\lambda_{d\frac{n}{n'}}(l_1 l_3 - l_1 l_5 - l_4 l_5 + l_3^2 - l_5^2) + \\ \qquad 2\gamma_{d\frac{n}{n'}} l_2^2 + 2\gamma_{m\frac{n}{n'}}(l_1^2 + l_2^2) + 2\gamma_{d'm'l}(-l_4^2) + \\ \qquad 2\gamma_{\frac{dm}{d'm'}l}(-l_1^2 - l_4^2) \ \textbf{(mod4)} \\ E_1 \equiv \lambda_{d'm'n'}(l_1 l_2 - l_2^2) + 2\lambda_{d'm'n'}(l_1 l_4 + l_1 l_6 + l_2 l_5) + 2\gamma_{d'm'n'} l_4^2 + \\ \qquad 2\gamma_{\frac{dm}{d'm'}n'}(l_1^2 + l_4^2) + 2\gamma_{\frac{dn}{d'n'}m'}(-l_2^2 - l_4^2) + \\ \qquad 2\gamma_{\frac{mn}{m'n'}d'}(-l_1^2 - l_2^2 - l_4^2) \ \textbf{(mod4)}, \end{cases}$$

and,

$$\begin{cases} C_1 \equiv \lambda_{dn'}(l_1 l_4 - l_4^2) + 2\lambda_{dn'}(-l_1 l_6 + l_2 l_3 - l_2 l_5 + l_4 l_5)(\textbf{mod4}) \\ D_1 \equiv \lambda_{d'm'l}(l_1 l_2 - l_2^2) + 2\lambda_{d'm'l}(l_2 l_3 - l_4 l_5)(\textbf{mod4}) \\ F_1 \equiv \lambda_{d'm'n'}(l_2^2 - l_1 l_2) + 2\lambda_{d'm'n'}(-l_1 l_6 - l_2 l_5)(\textbf{mod4}) \end{cases}$$

Resolution of the system (S_1) - Modular Calculations - Useful Lemmas

We are particularly interested in the system formed by the first three equations of (S_1).
We get:

$$\left(S_1'\right) : \begin{cases} A_1^2 - C_1^2 dm = 4\epsilon_1 \\ B_1^2 - D_1^2 dm = 4\epsilon_2 \\ E_1^2 - F_1^2 dm = 4\epsilon_3 \end{cases}, \tag{10}$$

Lemma

Let put:

$$\left\{ \begin{array}{l} d_1 = pgcd\left(A_1, C_1\right) \geq 1, \\ d_2 = pgcd\left(B_1, D_1\right) \geq 1, \\ d_3 = pgcd\left(E_1, F_1\right) \geq 1. \end{array} \right. \quad (11)$$

It is clear that (S_1') is solvable $\Longrightarrow d_1^2, d_2^2$ and d_3^2 divide $4 \Longrightarrow d_1,\ d_2,\ d_3 \in \{1, 2\}$.
We assume that (S_1') is solvable, so we have the following results.

We have the following three useful lemmas:
(a)

$$(S_0) : \begin{cases} \dfrac{n}{n'} A_1 = n' B_1 + l E_1 \\ \\ \dfrac{n}{n'} C_1 = n' D_1 + l F_1 \end{cases}$$

(b) The following system (S_1'') is solvable (exactly $(S_1') \Longleftrightarrow (S_1'')$).

$$(S_1''):\begin{cases} 2n'l \times \left[\dfrac{E_1B_1 - F_1D_1dm}{4}\right] = \left(\dfrac{n}{n'}\right)^2 \epsilon_1 - n'^2\epsilon_2 - l^2\epsilon_3 \\ 2\dfrac{n}{n'}l \times \left[\dfrac{A_1E_1 - C_1F_1dm}{4}\right] = \left(\dfrac{n}{n'}\right)^2 \epsilon_1 - n'^2\epsilon_2 + l^2\epsilon_3 \\ 2n \times \left[\dfrac{A_1B_1 - C_1D_1dm}{4}\right] = \left(\dfrac{n}{n'}\right)^2 \epsilon_1 + n'^2\epsilon_2 - l^2\epsilon_3 \end{cases} .$$

(c) $(d_1, d_2, d_3) = (2, 1, 1)$. So that
$A_1 \equiv C_1 \equiv 0 \textbf{ (mod2)}$, $B_1 \equiv D_1 \equiv 1 \textbf{ (mod2)}$
and $E_1 \equiv F_1 \equiv 1 \textbf{ (mod2)}$

(a) Let's establish the system (S_0):
Evident (by simple calculation).

(b) Let's show that (S_1') and (S_1'') are équivalent when (S_0) is checked.
Transforms (S_1') from the relations of (S_0).
We have:

$$A_1^2 - C_1^2 dm = 4\epsilon_1 \Longleftrightarrow \left(n' B_1 + l E_1\right)^2 - \left(n' D_1 + l F_1\right)^2 dm = \left(\frac{n}{n'}\right)^2 \times 4\epsilon_1$$

$$\Longleftrightarrow n'^2 \left(B_1^2 - D_1^2 dm\right) + l^2 \left(E_1^2 - F_1^2 dm\right) + 2n' l \left[E_1 B_1 - F_1 D_1 dm\right] = \left(\frac{n}{n'}\right)^2 \times 4\epsilon_1$$

$$\Longleftrightarrow n'^2 \times 4\varepsilon_1 + l^2 \times 4\varepsilon_3 + 2n' l \left[E_1 B_1 - F_1 D_1 dm\right] = \left(\frac{n}{n'}\right)^2 \times 4\epsilon_1.$$

Which gives us:

$$2n'l \times \left[\frac{E_1B_1 - F_1D_1dm}{4}\right] = \left(\frac{n}{n'}\right)^2\epsilon_1 - n'^2\epsilon_2 - l^2\epsilon_3.$$

Note that; $\left(\frac{n}{n'}\right)^2\epsilon_1 - n'^2\epsilon_2 - l^2\epsilon_3 \equiv \pm 1 \pmod 4$.

In the same way we have:

$B_1^2 - D_1^2dm = 4\epsilon_2$ equals to:

$$2\frac{n}{n'}l \times \left[\frac{A_1E_1 - C_1F_1dm}{4}\right] = \left(\frac{n}{n'}\right)^2\epsilon_1 - n'^2\epsilon_2 + l^2\epsilon_3$$

and $\left(\frac{n}{n'}\right)^2\epsilon_1 - n'^2\epsilon_2 - l^2\epsilon_3 \equiv \pm 1 \pmod{4}$.

Similarly; $E_1^2 - F_1^2 dm = 4\epsilon_1$ equals to:

$$2n \times \left[\frac{A_1B_1 - C_1D_1dm}{4}\right] = \left(\frac{n}{n'}\right)^2 \epsilon_1 + n'^2\epsilon_2 - l^2\epsilon_3.$$

And $\left(\frac{n}{n'}\right)^2 \epsilon_1 - n'^2\epsilon_2 - l^2\epsilon_3 \equiv \pm 1$ **(mod4)**.

The system (S_1') gives rise to the system (S_1'') below:

$$(S_1''): \begin{cases} 2n'l \times \left[\frac{E_1B_1 - F_1D_1dm}{4}\right] = \left(\frac{n}{n'}\right)^2 \epsilon_1 - n'^2\epsilon_2 - l^2\epsilon_3 \\ 2\frac{n}{n'}l \times \left[\frac{A_1E_1 - C_1F_1dm}{4}\right] = \left(\frac{n}{n'}\right)^2 \epsilon_1 - n'^2\epsilon_2 + l^2\epsilon_3 \\ 2n \times \left[\frac{A_1B_1 - C_1D_1dm}{4}\right] = \left(\frac{n}{n'}\right)^2 \epsilon_1 + n'^2\epsilon_2 - l^2\epsilon_3. \end{cases}.$$

For (c),
Let us first establish that: $(d_1, d_2,\ d_3) = (2, 1, 1)$ or $(1, 1, 2)$ or $(1, 2, 1)$.

• Suppose $d_1 = 2$, then A_1 and C_1 are even.

It is clear that the case $d_2 = d_3 = 2$ is impossible because otherwise $\dfrac{E_1 B_1 - F_1 D_1 dm}{4} \in \mathbb{Z} \implies 2n'l \times \left[\dfrac{E_1 B_1 - F_1 D_1 dm}{4}\right] = \left(\dfrac{n}{n'}\right)^2 \varepsilon_1 - n'^2 \varepsilon_2 - l^2 \varepsilon_3 \equiv 0 \pmod{2}$, which is absurd.

Similarly, if $(d_2,\ d_3) = (1,2)$ or $(2,1)$ then either B_1 and D_1 are even and E_1 and F_1 odd, or we have the opposite, in all cases as $\frac{n}{n'}A_1 = n'B_1 + lE_1$ and $\frac{n}{n'}C_1 = n'D_1 + lF_1$ it comes that A_1 and C_1 would be odd, which is absurd.
So the only possibility is $(d_1, d_2,\ d_3) = (2,1,1)$.

- Now suppose that $d_1 = 1$. So A_1 and C_1 are odd.

If we had $d_2 = d_3 = 2 \Longrightarrow B_1, D_1, E_1, F_1$ would be even $\Longrightarrow$ impossible, see above.
If we have $d_2 = d_3 = 1$, then B_1, D_1, E_1, F_1 would be odd, but then $(\frac{n}{n'}A_1 = n'B_1 + lE_1$ and $\frac{n}{n'}C_1 = n'D_1 + lF_1) \Longrightarrow A_1$ and C_1 would be even. This is absurd.

So $(d_1, d_2, d_3) = (1,1,2)$ or $(1,2,1)$.
In summary we have: $(d_1, d_2,\ d_3) = (2,1,1)$ *or* $(1,1,2)$ *or* $(1,2,1)$.
• To show that $(d_1, d_2,\ d_3) = (2,1,1)$, we use the computation of C_1, D_1, F_1 cf **proposition 2**, from which we deduce the value of $\frac{n}{n'}C_1 = n'D_1 + lF_1(\mathbf{mod4})$.
We get $\frac{n}{n'}C_1 \equiv n'D_1 + lF_1(\mathbf{mod4}) \Longrightarrow l_1 l_4 - l_4^2 \equiv 0(\mathbf{mod4})$.
As a consequence $C_1 \equiv 0$ **(mod2)** , so $A_1 \equiv 0$ **(mod2)** , because they are of the same parity.
So $d_1 = 2$ and consequently $(d_1, d_2,\ d_3) = (2,1,1)$ as announced in **lemma 1**.

Lemma

The system (S_1) being always supposed to be solvable, we also have the following Lemma:

(i) $l_1 \equiv l_4 \equiv 0 \textbf{ (mod2)}$; $l_3 \equiv 0 \textbf{(mod4)}$ and
$l_2 \equiv l_5 \equiv 1 \textbf{ (mod2)}$
(ii) For the quantities $A_1,\ B_1,\ E_1,\ C_1,\ D_1, F_1$ we have:

$$\begin{cases} C_1 \equiv -2\lambda_{dn'} l_2 l_5 \textbf{(mod4)}; \\ D_1 \equiv \lambda_{d'm'l}(l_1 l_2 - 1)\textbf{(mod4)}; \\ F_1 \equiv -\lambda_{d'm'n'}(l_1 l_2 - 1) - 2\lambda_{d'm'n'} l_2 l_5 \textbf{(mod4)}; \end{cases}$$

and

$$\begin{cases} A_1 \equiv -2\lambda_{dn'} + 2\lambda_{dn'} l_2 l_5 (\textbf{mod4}); \\ B_1 \equiv -\lambda_{d\frac{n}{n'}}(l_1 l_2 - 1) + 2\lambda_{\frac{n}{n'}}(\gamma_d + \gamma_m)(\textbf{mod4}); \\ E_1 \equiv \lambda_{d'm'n'}(l_1 l_2 - 1) + 2\lambda_{d'm'n'} l_2 l_5 - 2\lambda_{nl}(\gamma_d + \gamma_m)(\textbf{mod4}). \end{cases}$$

Remark

Note that the point (i) *of the lemma, is not sufficient to find a contradiction in computing modulo* 4, *the quantities* $(A_1, C_1), (B_1, D_1), (E_1, F_1), (G_1, H_1),$ $(I_1, J_1), (K_1, L_1), (M_1, N_1)$.

Proof

We had already calculated modulo 4, the quantities A_1, B_1, E_1 as well as C_1, D_1, F_1 cf **proposition 2**. Recall also that we have shown cf proof of **lemma 1** c) that $l_1 l_4 - l_4^2 \equiv 0$**(mod4)**
From which we deduce that F_1 is odd that: $l_2 \equiv 1$ **(mod2)** and $l_1 \equiv 0$ **(mod2)** and that accordingly $l_4 \equiv 0$ **(mod2)** .
Said congruences simplify a first time, considering among other things that $l_2^2 \equiv 1$ **(mod4)**, we find:

Proof

$$\begin{cases} A_1 \equiv 2\lambda_{dn'}(-l_2 l_3 + l_2 l_5 + l_3^2 - l_5^2) + \\ 2(\gamma_{dn'} + \gamma_{mn'} - \gamma_{\frac{d}{d'} m' l} - \gamma_{\frac{m}{m'} d' l}) \ \mathbf{(mod 4)} \\ B_1 \equiv \lambda_{d \frac{n}{n'}}(-l_1 l_2 + 1) + 2\lambda_{d \frac{n}{n'}}(l_3^2 - l_5^2) + 2(\gamma_{d \frac{n}{n'}} + \gamma_{m \frac{n}{n'}}) \ \mathbf{(mod 4)} \\ E_1 \equiv \lambda_{d' m' n'}(l_1 l_2 - 1) + 2\lambda_{d' m' n'}(l_2 l_5) - 2(\gamma_{\frac{dn}{d' n'} m'} + \gamma_{\frac{mn}{m' n'} d'}) \ \mathbf{(mod 4)} \end{cases}$$

and,

Proof

$$\begin{cases} C_1 \equiv 2\lambda_{dn'}(l_2 l_3 - l_2 l_5)\textbf{(mod4)} \\ D_1 \equiv \lambda_{d'm'l}(l_1 l_2 - 1) + 2\lambda_{d'm'l}(l_2 l_3)\textbf{(mod4)} \\ F_1 \equiv \lambda_{d'm'n'}(1 - l_1 l_2) + 2\lambda_{d'm'n'}(-l_2 l_5)\textbf{(mod4)} \end{cases}$$

• Now we consider: $\frac{n}{n'}A_1 \equiv n'B_1 + lE_1$ **(mod4)**, (see **lemma 1** (a)) considering that $\lambda_{dn} = \lambda_{d'm'n'l} = 1$, gives the reduction: $-l_2 l_3 \equiv -2\lambda_{\frac{n}{n'}}(\gamma_{dn'} + \gamma_{mn'} - \gamma_{\frac{d}{d'}m'l} - \gamma_{\frac{m}{m'}d'l}) + 2(\lambda_{n'}(\gamma_{d\frac{n}{n'}} + \gamma_{m\frac{n}{n'}}) - \lambda_l(\gamma_{\frac{dn}{d'n'}m'} + \gamma_{\frac{mn}{m'n'}d'}))$ **(mod4)**.

Proof

But the two quantities:

$2(\gamma_{dn'} + \gamma_{mn'} - \gamma_{\frac{d}{d'}m'l} - \gamma_{\frac{m}{m'}d'l})$ and

$2(\lambda_{n'}(\gamma_{d\frac{n}{n'}} + \gamma_{m\frac{n}{n'}}) - \lambda_l(\gamma_{\frac{dn}{d'n'}m'} + \gamma_{\frac{mn}{m'n'}d'}))$ are $\equiv 0(\textbf{mod4})$.

Indeed, let's use the form of **remark 1**

- $$2(\gamma_{dn'} + \gamma_{mn'} - \gamma_{\frac{d}{d'}m'l} - \gamma_{\frac{m}{m'}d'l}) \equiv 2(\lambda_d\gamma_{n'} + \lambda_{n'}\gamma_d + \lambda_m\gamma_{n'} + \lambda_{n'}\gamma_m - \lambda_{\frac{d}{d'}m'}\gamma_l - \lambda_l\gamma_{\frac{d}{d'}m'} - \lambda_{\frac{m}{m'}d'}\gamma_l - \lambda_l\gamma_{\frac{m}{m'}d'})(\textbf{mod4})$$
  $$\equiv 2(\gamma_{n'}(\lambda_d + \lambda_m) + \lambda_{n'}(\gamma_d + \gamma_m) - \gamma_l(\lambda_{\frac{d}{d'}m'} + \lambda_{\frac{m}{m'}d'}) - \lambda_l(\gamma_{\frac{d}{d'}m'} + \gamma_{\frac{m}{m'}d'}))(\textbf{mod4})$$
  $$\equiv 2(\lambda_{n'}(\gamma_d + \gamma_m) - \lambda_l(\gamma_{\frac{d}{d'}m'} + \gamma_{\frac{m}{m'}d'}))(\textbf{mod4})$$
  $$\equiv 2(\lambda_{n'}(\lambda_{d'}\gamma_{\frac{d}{d'}} + \lambda_{\frac{d}{d'}}\gamma_d + \lambda_{m'}\gamma_{\frac{m}{m'}} + \lambda_{\frac{m}{m'}}\gamma_{m'}) - \lambda_l(\lambda_{\frac{d}{d'}}\gamma_{m'} + \lambda_{m'}\gamma_{\frac{d}{d'}} + \lambda_{\frac{m}{m'}}\gamma_{d'} + \lambda_{d'}\gamma_{\frac{m}{m'}}))(\textbf{mod4})$$
  $$\equiv 2(\lambda_{d'n'}\gamma_{\frac{d}{d'}} + \lambda_{\frac{d}{d'}n'}\gamma_{d'} + \lambda_{m'n'}\gamma_{\frac{m}{m'}} + \lambda_{\frac{m}{m'}n'}\gamma_{m'} - (\lambda_{\frac{d}{d'}l}\gamma_{m'} + \lambda_{m'l}\gamma_{\frac{d}{d'}} + \lambda_{\frac{m}{m'}}\gamma_{d'} + \lambda_{d'l}\gamma_{\frac{m}{m'}}))(\textbf{mod4})$$

Proof

$$\equiv 2(\gamma_{\frac{d}{d'}}(\lambda_{d'n'} - \lambda_{m'l}) + \gamma_{d'}(\lambda_{\frac{d}{d'}n'} - \lambda_{\frac{m}{m'}l}) + \gamma_{\frac{m}{m'}}(\lambda_{m'n'} - \lambda_{d'l}) + \gamma_{m'}(\lambda_{\frac{m}{m'}n'} - \lambda_{\frac{d}{d'}l}))\textbf{(mod4)}$$
$$\equiv 0\textbf{(mod4)}.$$

Proof

Similarly, we have:

$2(\gamma_{d\frac{n}{n'}} + \gamma_{m\frac{n}{n'}}) \equiv 2\lambda_{\frac{n}{n'}}(\gamma_d + \gamma_m))\textbf{(mod4)}$

$\equiv 2\lambda_{\frac{n}{n'}}(\gamma_d + \gamma_m))\textbf{(mod4)}$

$\equiv 2\lambda_{\frac{n}{n'}}(\lambda_{\frac{d}{d'}}\gamma_{d'} + \lambda_{d'}\gamma_{\frac{d}{d'}} + \lambda_{\frac{m}{m'}}\gamma_{m'} + \lambda_{m'}\gamma_{\frac{m}{m'}})\textbf{(mod4)}$

and,

Proof

$$2(\gamma_{\frac{dn}{d'n'}m'} + \gamma_{\frac{mn}{m'n'}d'}) \equiv 2\lambda_{\frac{n}{n'}}(\gamma_{\frac{d}{d'}m'} + \gamma_{\frac{m}{m'}d'}))\textbf{(mod4)}$$
$$\equiv 2\lambda_{\frac{n}{n'}}(\lambda_{\frac{m}{m'}}\gamma_{d'} + \lambda_{m'}\gamma_{\frac{d}{d'}} + \lambda_{\frac{d}{d'}}\gamma_{m'} + \lambda_{d'}\gamma_{\frac{m}{m'}})\textbf{(mod4)}$$
So $2(\lambda_{n'}(\gamma_{d\frac{n}{n'}} + \gamma_{m\frac{n}{n'}}) - \lambda_l(\gamma_{\frac{dn}{d'n'}m'} + \gamma_{\frac{mn}{m'n'}d'})) \equiv$
$2(\lambda_{n'}\lambda_{\frac{n}{n'}}(\gamma_d + \gamma_m) - \lambda_l\lambda_{\frac{n}{n'}}(\gamma_{\frac{d}{d'}m'} + \gamma_{\frac{m}{m'}d'})) \equiv$
$2(\gamma_{d'}(\lambda_{\frac{d}{d'}n} - \lambda_{\frac{m}{m'}\frac{n}{n'}l}) + \gamma_{\frac{d}{d'}}(\lambda_{d'n} - \lambda_{m'\frac{n}{n'}l}) + \gamma_{m'}(\lambda_{\frac{m}{m'}n} - \lambda_{\frac{d}{d'}\frac{n}{n'}l}) +$
$\gamma_{\frac{m}{m'}}(\lambda_{m'n} - \lambda_{\frac{d}{d'}\frac{n}{n'}l}) \equiv$
$0\textbf{(mod4)}$.

Proof

This implies that $-l_2 l_3 \equiv 0(\mathbf{mod 4}) \Longrightarrow l_3 \equiv 0(\mathbf{mod 4})$:
But then $l_5 \equiv 1$ **(mod2 (** because otherwise we would have $d_1 = 4$, which would be absurd.
The congruences are re-written in finality as follows, noting the fact that
$2(\gamma_{\frac{dn}{d'n'}m'} + \gamma_{\frac{mn}{m'n'}d'}) \equiv 2\lambda_{n'l}(\gamma_{d\frac{n}{n'}} + \gamma_{m\frac{n}{n'}}) \equiv 2\lambda_{nl}(\gamma_d + \gamma_m)(\mathbf{mod 4})$
what makes it possible to make appear this quantity in the expression of E_1 at the level of the Lemma.
And we have as well as announced:

Proof

$$\begin{cases} C_1 \equiv -2\lambda_{dn'} l_2 l_5 \textbf{(mod4)} \\ D_1 \equiv \lambda_{d'm'l}(l_1 l_2 - 1)\textbf{(mod4)} \\ F_1 \equiv -\lambda_{d'm'n'}(l_1 l_2 - 1) - 2\lambda_{d'm'n'} l_2 l_5 \textbf{(mod4)} \end{cases}$$

and

$$\begin{cases} A_1 \equiv -2\lambda_{dn'} + 2\lambda_{dn'} l_2 l_5 \textbf{(mod4)} \\ B_1 \equiv -\lambda_{d\frac{n}{n'}}(l_1 l_2 - 1) + 2(\gamma_d + \gamma_m)\textbf{(mod4)} \\ E_1 \equiv \lambda_{d'm'n'}(l_1 l_2 - 1) + 2\lambda_{d'm'n'} l_2 l_5 - 2\lambda_{nl}(\gamma_d + \gamma_m)\textbf{(mod4)}. \end{cases}$$

Non-monogeneity of the fields $K_3 = \mathbb{Q}(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l})$ with odd discriminant

We are now equipped to give the main theorem **theorem (2)** of this article.

Theorem

Let be a triquadratic field $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ *with odd discriminant, ie such that* $(dm, dn, d'm'n'l) \equiv (1, 1, 1)$ **(mod4)** *Then* K_3 *is not monogenous ie the system* (S_1) *associated with the equation of monogeneity* **(8)** *of* K_3 *is not solvable.*

proof

To proof this Theorem, it suffices to show that the system (S_1) admits no solution. Indeed the conditions of **lemma 1** are strong enough to show that the system (S_1) is not solvable because (S_1'') (ie (S_1')) is not. We will show that in the first equation of (S_1') cf **lemma 1**(b), we have:

$$E_1B_1 - F_1D_1dm \equiv 0(\mathbf{mod4}) \Longrightarrow \left(\frac{n}{n'}\right)^2 \epsilon_1 - n'^2\epsilon_2 - l^2\epsilon_3 \equiv 0 \pmod{\mathbf{2}},$$

which is absurd since all summed numbers are odd.
To show this, let's calculate $(E_1B_1 - F_1D_1dm)\,(\mathbf{mod4})$.
Using the last **lemma 2**(ii), we have:
$E_1B_1 - F_1D_1dm \equiv E_1B_1 - F_1D_1 \equiv$
$(\lambda_{d'm'n'}(l_1l_2 - 1) + 2\lambda_{d'm'n'}l_2l_5 - 2\lambda_{nl}(\gamma_d + \gamma_m))\times$
$(-\lambda_{d\frac{n}{n'}}(l_1l_2 - 1) + 2(\gamma_d + \gamma_m)) - (-\lambda_{d'm'n'}(l_1l_2 - 1) -$
$2\lambda_{d'm'n'}l_2l_5) \times (\lambda_{d'm'l}(l_1l_2 - 1))(\mathbf{mod4})$.

So that

$$E_1 B_1 - F_1 D_1 dm \equiv$$
$$(\lambda_{n'l} - \lambda_{\frac{d}{d'}m'n}) + 2(l_1 l_2 - 1)(\gamma_d + \gamma_m)(\lambda_{d'm'n'} + \lambda_{dn'l}) -$$
$$2\lambda_{\frac{d}{d'}m'n} l_2 l_5 (l_1 l_2 - 1)(\lambda_{n'l} - \lambda_{\frac{d}{d'}m'n}) \equiv 0 (\mathbf{mod 4}). \blacksquare$$

Conclusion

If $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ *with* $(dm, dn, d'm'n'l) \equiv (1, 1, 1)$ **(mod4)**, *then the monogeneity equation does not admit solutions in* $\mathbb{Z}_{K_3}$. *It means that:*

"Any triquadratic number field with odd discriminant ismonogenous."

We find the result of Y. MOTODA and T. NAKAHARA [7], they used the ramification of 2. G. NYUL [9] also got this result by using the index form and the indices of sub-groups.
This method of proof should be able to apply a priori when the discriminant is even; that is to say the two remaining cases: $(dm, dn, d'm'n'l) \equiv (1, 1, 2 \text{ or } 3)$ and $\equiv (1, 2, 3)$ **(mod4)**. Indeed in each of these cases, similar lemmas to those used here have been establshed, and we were able to conclude (cf [4]), (cf comments paragraph).

Comments

We give here the elements necessary for a Diophantine proof, similar to the previous one, of the non-monogeneity of triquadatic fields, in the remaining cases 2 and 3. However, note that the cyclotomic body of degree 8, $\mathbb{Q}(\zeta_{24}) = \mathbb{Q}\left(\sqrt{-3}, \sqrt{2}, \sqrt{-1}\right)$ which belongs to case 3, is the only triquadratic body which is monogenic. We treat an example of an integral K_3-Signal.
Both in case 2 and case 3, we will use the results of [4] [5] , [13], let the following theorem

Theorem 1.1

We have the following theorem:

Let $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ with $(dm, dn, d'm'n'l) \equiv (1, 1, 2\ or\ 3) \pmod 4$. (That means we are in case 2).

(i) A Chatelain's written form of $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ is:

$$K_3 = \begin{cases} \mathbb{Q}\left(\sqrt{dm},\ \sqrt{dn}, \sqrt{2\lambda_{d'm'n'\frac{l}{2}}}\sqrt{\lambda_{d'm'n'\frac{l}{2}}\ d'm'n'\frac{l}{2}}\right), & \text{if } l \text{ is even} \\ \mathbb{Q}\left(\sqrt{dm},\ \sqrt{dn}, \sqrt{-1}\sqrt{-d'm'n'l}\right), & \text{either} \end{cases}$$

(ii) The corresponding Chatelain's $\beta-$basis of K_3 is given by:

$$\beta = \left\{1, \sqrt{dm}, \sqrt{dn}, s_1\sqrt{mn}, \gamma\sqrt{d'm'n'l}, \gamma s_2\sqrt{\frac{d}{d'}\frac{m}{m'}n'l}, \gamma s_3\sqrt{\frac{d}{d'}\frac{n}{n'}m'l},\right.$$
$$\left.\gamma s_4\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}\right\}.$$

Theorem 1.1

(iii) The corresponding Chatelain's $\mathbb{Z}-$base $\mathfrak{B}_{K_3}$ of $\mathbb{Z}_{K_3}$ is the following:

$$\varepsilon_0 = \frac{1+\sqrt{dm}+\sqrt{dn}+s_1\sqrt{mn}}{4} \ ,$$

$$\varepsilon_1 = \frac{1-\sqrt{dm}+\sqrt{dn}-s_1\sqrt{mn}}{4},$$

$$\varepsilon_2 = \frac{1+\sqrt{dm}-\sqrt{dn}-s_1\sqrt{mn}}{4},$$

$$\varepsilon_3 = \frac{1-\sqrt{dm}-\sqrt{dn}+s_1\sqrt{mn}}{4},$$

$$\varepsilon_4 = \gamma\frac{\sqrt{d'm'n'l}+s_2\sqrt{\frac{d}{d'}\frac{m}{m'}n'l}+s_3\sqrt{\frac{d}{d'}\frac{n}{n'}m'l}+s_4\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}}{4},$$

$$\varepsilon_5 = \gamma\frac{\sqrt{d'm'n'l}-s_2\sqrt{\frac{d}{d'}\frac{m}{m'}n'l}+s_3\sqrt{\frac{d}{d'}\frac{n}{n'}m'l}-s_4\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}}{4},$$

$$\varepsilon_6 = \gamma\frac{\sqrt{d'm'n'l}+s_2\sqrt{\frac{d}{d'}\frac{m}{m'}n'l}-s_3\sqrt{\frac{d}{d'}\frac{n}{n'}m'l}-s_4\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}}{4},$$

Theorem 1.2

$$\varepsilon'_0 = \frac{1+\sqrt{dm}+\sqrt{dn}+s_1\sqrt{mn}}{4}\ ;\ \varepsilon'_1 = \frac{-\sqrt{dm}-\sqrt{dn}}{2}\ ;\ \varepsilon'_2 = \frac{-\sqrt{dn}-s_1\sqrt{mn}}{2};$$

$$\varepsilon'_3 = -s_1\sqrt{mn}\ ;$$

$$\varepsilon'_4 = \gamma\frac{\sqrt{d'm'n'l}+s_2\sqrt{\frac{dm}{d'm'}n'l}+s_3\sqrt{\frac{dn}{d'n'}m'l}+s_4\sqrt{\frac{mn}{m'n'}d'l}}{4};$$

$$\varepsilon'_5 = \frac{-\gamma}{2}\left(s_2\sqrt{\frac{dm}{d'm'}n'l}+s_3\sqrt{\frac{dn}{d'n'}m'l}\right)\ ;$$

$$\varepsilon'_6 = \frac{-\gamma}{2}\left(s_3\sqrt{\frac{dn}{d'n'}m'l}+s_4\sqrt{\frac{mn}{m'n'}d'l}\right)\ ;$$

$$\varepsilon'_7 = -\gamma s_4\sqrt{\frac{mn}{m'n'}d'l}\ .$$

Theorem 1.3

In case 2, the integral elements are characterized as follows:
Let $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$, a triquadratic number field, belonging to case 2. There is equivalencies between propositions (i) and (ii).
(i) $\theta = \omega_0 + \omega_1\sqrt{dm} + \omega_2\sqrt{dn} + \omega_3\sqrt{mn} + \omega_4\sqrt{d'm'n'l} + \omega_5\sqrt{\frac{dm}{d'm'}n'l} + \omega_6\sqrt{\frac{dn}{d'n'}m'l} + \omega_7\sqrt{\frac{mn}{m'n'}d'l} \in \mathbb{Z}_{K_3}$, (where $\omega_i \in \mathbb{Q}$).

Theorem 1.3

(ii) $\exists\ (a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \in \mathbb{Z}^8$, such that:
$B_1)$ $\theta = \frac{1}{4}(a_0 + a_1\sqrt{dm} + a_2\sqrt{dn} + s_1 a_3\sqrt{mn} + \gamma a_4\sqrt{d'm'n'l} +$
$\gamma s_2 a_5\sqrt{\frac{dm}{d'm'}n'l} + \gamma s_3 a_6\sqrt{\frac{dn}{d'n'}m'l} + \gamma s_4 a_7\sqrt{\frac{mn}{m'n'}d'l})$, and
satisfying:

$$B_2)\quad \begin{cases} a_0 \in \mathbb{Z} \\ a_0 - a_1 \equiv 0 \pmod 2 \\ a_1 - a_2 \equiv 0 \pmod 2 \\ a_0 - a_1 + a_2 - a_3 \equiv 0 \pmod 4 \\ a_4 \in \mathbb{Z} \\ a_4 - a_5 \equiv 0 \pmod 2 \\ a_5 - a_6 \equiv 0 \pmod 2 \\ a_4 - a_5 + a_6 - a_7 \equiv 0 \pmod 4 \end{cases}$$

Now we have:

$$\mathbb{Z}_{K_3} \ni \theta = \frac{1}{4}(a_0 + a_1\sqrt{dm} + a_2\sqrt{dn} + s_1 a_3\sqrt{mn} + \gamma a_4\sqrt{d'm'n'l} + \gamma s_2 a_5\sqrt{\frac{dm}{d'm'}n'l} + \gamma s_3 a_6\sqrt{\frac{dn}{d'n'}m'l} + \gamma s_4 a_7\sqrt{\frac{mn}{m'n'}d'l})$$

$$= l_0\varepsilon'_0 + l_1\varepsilon'_1 + l_2\varepsilon'_2 + l_3\varepsilon'_3 + l_4\varepsilon'_4 + l_5\varepsilon'_5 + l_6\varepsilon'_7 + l_6\varepsilon'_7.$$

Where:

$$\begin{cases} l_0 &= a_0 \in \mathbb{Z} \\ l_1 &= \dfrac{a_0 - a_1}{2} \in \mathbb{Z} \\ l_2 &= \dfrac{a_0 - a_2}{2} \in \mathbb{Z} \\ l_3 &= \dfrac{a_0 + a_1 - a_2 - a_3}{4} \in \mathbb{Z} \\ l_4 &= \dfrac{a_0 - a_4}{2} \in \mathbb{Z} \end{cases} ; \begin{cases} l_5 &= \dfrac{a_0 - a_1 - a_4 + a_5}{4} \in \mathbb{Z} \\ l_6 &= \dfrac{a_0 + a_2 - a_4 - a_6}{4} \in \mathbb{Z} \end{cases}$$

$$l_7 = \frac{a_0 + a_1 + a_2 + a_3 - a_4 - a_5 - a_6 - a_7}{8} \in \mathbb{Z}$$

From this, the monogeneity equations and lemmas corresponding to (8) are:

$$(S_2) : \begin{cases} \Delta_2(\theta) = s_1 \\ \Delta_6(\theta) = s_2 \\ \left(\frac{1}{4}\right)^2 \Delta_4(\theta) = s_3 \\ \Delta_1(\theta) = s_4 \\ \Delta_5(\theta) = s_5 \\ \Delta_7(\theta) = s_6 \\ \Delta_3(\theta) = s_7 \end{cases}$$

Where $s_k = \pm 1,\ k = 1, ..., 7$.
The first 3 equations sufficient to solve the problem are:

$$\left(S_2'\right): \begin{cases} A_2^2 - C_2^2 dm = 4s_1 \\ B_2^2 - D_2^2 dm = 4s_2 \\ E_2^2 - F_2^2 dm = 4s_3 \end{cases}$$

Where:

$$A_2 = \frac{a_2^2 dn' + a_3^2 mn' - a_6^2 \frac{d}{d'} m' l - a_7^2 \frac{m}{m'} d' l}{2},$$

$$C_2 = a_2 a_3 s_1 n' - a_6 a_7 s_3 s_4 l,$$

$$B_2 = \frac{a_2^2 d \frac{n}{n'} + a_3^2 m \frac{n}{n'} - a_4^2 d' m' l - a_5^2 \frac{dm}{d' m'} l}{2},$$

$$D_2 = a_2 a_3 s_1 \frac{n}{n'} - a_4 a_5 s_2 l,$$

$$E_2 = \frac{a_4^2 d' m' n' + a_5^2 \frac{dm}{d' m'} n' - a_6^2 \frac{dn}{d' n'} m' - a_7^2 \frac{mn}{m' n'} d'}{2 \times 4},$$

$$F_2 = \frac{a_4 a_5 s_2 n' - a_6 a_7 s_3 s_4 \frac{n}{n'}}{4}.$$

Then A_2, B_2, C_2, D_2, E_2 and $F_2 \in \mathbb{Z}$, and $(A_2, C_2), (B_2, D_2)$ and (E_2, F_2) consist of integers with the same parity.

Lemmas useful in the case $(dm, dn, d'm'n'l) \equiv (1, 1, 2 \text{ ou } -1) \pmod 4$

1) (a) We get: $\begin{cases} \left(\dfrac{n}{n'}\right) A_2 = n' B_2 + 4l.E_2 \\ \\ \left(\dfrac{n}{n'}\right) C_2 = n' D_2 + 4l.F_2 \end{cases}$

(b) The systems below are equivalent.

$$(S_2') : \begin{cases} A_2^2 - C_2^2 dm = 4s_1 \\ B_2^2 - D_2^2 dm = 4s_2 \\ E_2^2 - F_2^2 dm = 4s_3 \end{cases} \quad \text{and}$$

$$(S_2'') : \begin{cases} 2n'l \times [E_2 B_2 - F_2 D_2 dm] = \left(\dfrac{n}{n'}\right)^2 s_1 - n'^2 s_2 - (4l)^2 s_3 \\ 2\dfrac{n}{n'} l \times [A_2 E_2 - C_2 F_2 dm] = \left(\dfrac{n}{n'}\right)^2 s_1 - n'^2 s_2 + (4l)^2 s_3 \\ 2n \times \left[\dfrac{A_2 B_2 - C_2 D_2 dm}{4}\right] = \left(\dfrac{n}{n'}\right)^2 s_1 + n'^2 s_2 - (4l)^2 s_3 \end{cases}$$

Lemma

(c) On a: $s_1 = s_2 = s$.
2)(a) A_2, B_2, C_2 and D_2 are odd.
$(b) s_3 s_4 = s_1 s_2 = 1 \Rightarrow s_3 = s_4 = s$.

$$(c) F_2 \equiv \begin{cases} \frac{\lambda_{n'} a_4 a_5 s_2}{4} \times (1 - \lambda_d) \pmod 2 \text{ si } \lambda_{n'} = \lambda_{\frac{n}{n'}} \\ \frac{\lambda_{n'} a_4 a_5 s_2}{4} \times (1 + \lambda_d) \pmod 2 \text{ si } \lambda_{n'} = -\lambda_{\frac{n}{n'}} \end{cases}$$

Now replace the a_i by this expressions in function of the l_i, then the demonstration is carried out in the same way, and we end up with similar contradictions $\pmod 4$.

This time, we'll found that the only one triquadratic field, to be monogenic is the cyclotomic one: $\mathbb{Q}(\zeta_{24}) = \mathbb{Q}\left(\sqrt{-3}, \sqrt{2}, \sqrt{-1}\right)$. We get this result about the $\mathbb{Z} - base$.

Theorem 1.4

For **Case 3**: $(dm, dn, d'm'n'l) \equiv (1, 2, 3) \pmod 4$

(i) A Chatelain's written form of $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ is:

$$K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{2\delta}\sqrt{\delta\ d\frac{n}{2}}, \sqrt{-1}\sqrt{-d'm'n'l}\right) \ .$$

(ii) The corresponding Chatelain's $\beta-$basis of K_3 is given by:

$$\beta = \left\{1, \sqrt{dm}, \delta\sqrt{dn}, \delta s_1\sqrt{mn}, -\sqrt{d'm'n'l}, -s_2\sqrt{\frac{d}{d'}\frac{m}{m'}n'l}, -\delta s_3\sqrt{\frac{d}{d'}\frac{n}{n'}m'l}, -\delta s_4\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}\right\}.$$

Theorem 1.4

(iii) The corresponding de Chatelain's $\mathbb{Z}-$base $\mathfrak{B}_{K_3}$ of $\mathbb{Z}_{K_3}$ is the following:

$$\varepsilon_0 = \frac{1+\sqrt{dm}}{2}, \quad \varepsilon_1 = \delta\frac{\sqrt{dn}+s_1\sqrt{mn}}{2},$$

$$\varepsilon_2 = \frac{-\sqrt{d'm'n'l} - s_2\sqrt{\frac{d}{d'}\frac{m}{m'}n'l}}{2},$$

$$\varepsilon_3 = \delta\frac{\sqrt{dn}+s_1\sqrt{mn} - s_3\sqrt{\frac{d}{d'}\frac{n}{n'}m'l} - s_4\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}}{4},$$

$$\varepsilon_4 = \frac{1-\sqrt{dm}}{2}, \quad \varepsilon_5 = \delta\frac{\sqrt{dn}-s_1\sqrt{mn}}{2},$$

$$\varepsilon_6 = \frac{-\sqrt{d'm'n'l} + s_2\sqrt{\frac{d}{d'}\frac{m}{m'}n'l}}{2} \quad \text{and}$$

$$\varepsilon_7 = \delta\frac{\sqrt{dn}-s_1\sqrt{mn} - s_3\sqrt{\frac{d}{d'}\frac{n}{n'}m'l} + s_4\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}}{4}.$$

The discriminant is $\mathfrak{D}_{K_3/\mathbb{Q}} = \left(2^3 dmnl\right)^4$

Lemma 1.2

Let's recall that in this case , there is too a particular suitable and useful integral bases $\mathfrak{B}'_{K_3} = \{\varepsilon'_0, \varepsilon'_1, \varepsilon'_2, \varepsilon'_3, \varepsilon'_4, \varepsilon'_5, \varepsilon'_6, \varepsilon'_7\}$ obtained from the previous $\mathfrak{B}_{K_3}$ ones, and we will use in the proof :

Lemma

3) For **Case 3***:*

$$\varepsilon'_0 = \frac{1+\sqrt{dm}}{2}, \ \ \varepsilon'_1 = \sqrt{dm},$$

$$\varepsilon'_2 = \frac{\delta}{4}\left(\sqrt{dn} + s_1\sqrt{mn} - s_3\sqrt{\tfrac{dn}{d'n'}m'l} - s_4\sqrt{\tfrac{mn}{m'n'}d'l}\right),$$

$$\varepsilon'_3 = \frac{\delta}{2}\left(s_1\sqrt{mn} - s_4\sqrt{\tfrac{mn}{m'n'}d'l}\right),$$

$$\varepsilon'_4 = \frac{1}{2}\left(-\sqrt{d'm'n'l} - s_2\sqrt{\tfrac{dm}{d'm'}n'l}\right),$$

$$\varepsilon'_5 = \frac{1}{4}\left(-s_2\sqrt{\tfrac{dm}{d'm'}n'l}\right), \ \ \varepsilon'_6 = \frac{\delta}{2}\left(-s_3\sqrt{\tfrac{dn}{d'n'}m'l} - s_4\sqrt{\tfrac{mn}{m'n'}d'l}\right),$$

$$\varepsilon'_7 = \frac{\delta}{2}s_4\sqrt{\tfrac{mn}{m'n'}d'l} \ .$$

Theorem 1.5

The characterization of any elements of $\mathbb{Z}_{K_3}$, is the following one: Let $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$, a triquadratic number field, belonging to case 3. Then there is equivalencies between propositions (i) and (ii).

(i)

$$\theta = \omega_0 + \omega_1\sqrt{dm} + \omega_2\sqrt{dn} + \omega_3\sqrt{mn} + \omega_4\sqrt{d'm'n'l} + \omega_5\sqrt{\frac{dm}{d'm'}n'l} + \omega_6\sqrt{\frac{dn}{d'n'}m'l} + \omega_7\sqrt{\frac{mn}{m'n'}d'l} \in \mathbb{Z}_{K_3}\ (\omega_i \in \mathbb{Q}).$$

(ii) $\exists\ (a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \in \mathbb{Z}^8$, such that:

$C_1)$ $\theta = \frac{1}{4}(a_0 + a_1\sqrt{dm} + \delta a_2\sqrt{dn} + \delta s_1 a_3\sqrt{mn} - a_4\sqrt{d'm'n'l} - s_2 a_5\sqrt{\frac{dm}{d'm'}n'l} - \delta s_3 a_6\sqrt{\frac{dn}{d'n'}m'l} - \delta s_4 a_7\sqrt{\frac{mn}{m'n'}d'l})$, and satisfying:

Theorem 1.5

$$C_2) \quad \begin{cases} a_0 \equiv 0 \pmod{2} \\ a_1 - a_2 \equiv 0 \pmod{4} \\ a_2 \in \mathbb{Z} \\ a_3 - a_2 \equiv 0 \pmod{2} \\ a_4 \equiv 0 \pmod{2} \\ a_5 - a_4 \equiv 0 \pmod{4} \\ a_6 - a_2 \equiv 0 \pmod{2} \\ a_2 - a_3 - a_6 + a_7 \equiv 0 \pmod{2} \end{cases}$$

Note that:

$$\mathbb{Z}_{K_3} \ni \theta = \frac{1}{4}(a_0+a_1\sqrt{dm}+\delta a_2\sqrt{dn}+\delta s_1 a_3\sqrt{mn}-a_4\sqrt{d'm'n'l}- s_2 a_5\sqrt{\frac{dm}{d'm'}n'l}-\delta s_3 a_6\sqrt{\frac{dn}{d'n'}m'l}-\delta s_4 a_7\sqrt{\frac{mn}{m'n'}d'l})$$

$$= l_0\varepsilon'_0 + l_1\varepsilon'_1 + l_2\varepsilon'_2 + l_3\varepsilon'_3 + l_4\varepsilon'_4 + l_5\varepsilon'_5 + l_6\varepsilon'_6 + l_7\varepsilon'_7.$$

$$\text{Where: } \begin{cases} a_0 = 2l_0 \in 2\mathbb{Z} \\ \dfrac{a_1 - a_0}{4} = l_1 \in \mathbb{Z} \\ a_2 = l_2 \in \mathbb{Z} \\ \dfrac{a_3 - a_2}{2} = l_3 \in \mathbb{Z} \\ a_4 = 2l_4 \in 2\mathbb{Z} \\ \dfrac{a_5 - a_4}{4} = l_5 \in \mathbb{Z} \\ \dfrac{a_6 - a_2}{2} = l_6 \in \mathbb{Z} \\ \dfrac{a_2 - a_3 - a_6 + a_7}{2} = l_7 \in \mathbb{Z} \end{cases}$$

Lemmas useful in the case $(dm, dn, d'm'n'l) \equiv (1, 2, 3) \pmod 4$

In this case also we have a certain number of lemmas which should allow us to conclude. The monogeneity equation is equivalent to the following system:

$$(S_3) \quad : \quad \begin{cases} \frac{1}{4}\Delta_2(\theta) = s_1 \\ \frac{1}{4}\Delta_6(\theta) = s_2 \\ \frac{1}{4}\Delta_4(\theta) = s_3 \\ \Delta_5(\theta) = s_4 \\ \Delta_1(\theta) = s_5 \\ \Delta_7(\theta) = s_6 \\ \Delta_3(\theta) = s_7 \end{cases}$$

$$\text{where } s_k = \pm 1, k = 1, ..., 7.$$

Whose first 3 equations are:

Where:

$$
\begin{aligned}
A_3 &= \frac{a_2^2 d n' + a_3^2 m n' - a_6^2 \frac{d}{d'} m' l - a_7^2 \frac{m}{m'} d' l}{4}, \\
C_3 &= \frac{a_2 a_3 s_1 n' - a_6 a_7 s_3 s_4 l}{2}, \\
B_3 &= \frac{a_2^2 d \frac{n}{n'} + a_3^2 m \frac{n}{n'} - a_4^2 d' m' l - a_5^2 \frac{dm}{d' m'} l}{4}, \\
D_3 &= \frac{a_2 a_3 s_1 \frac{n}{n'} - a_4 a_5 s_2 l}{2}, \\
E_3 &= \frac{a_4^2 d' m' n' + a_5^2 \frac{dm}{d' m'} n' - a_6^2 \frac{dn}{d' n'} m' - a_7^2 \frac{mn}{m' n'} d'}{4}, \\
F_3 &= \frac{a_4 a_5 s_2 n' - a_6 a_7 s_3 s_4 \frac{n}{n'}}{2}.
\end{aligned}
$$

Lemma 1.3

1) (*a*) We have:

$$\begin{cases} \left(\dfrac{n}{n'}\right) A_3 = lE_3 + n' B_3 \\ \\ \left(\dfrac{n}{n'}\right) C_3 = lF_3 + n' D_3 \end{cases} \tag{3.2}$$

Lemma 1.3

(b) The systems below are equivalent:

$$(S_3') \quad : \quad \begin{cases} A_3^2 - C_3^2 dm = 4s_1 \\ B_3^2 - D_3^2 dm = 4s_2 \quad \text{et} \\ E_3^2 - F_3^2 dm = 4s_3 \end{cases}$$

$$(S_3'') \quad : \quad \begin{cases} 2n'l \times \left[\dfrac{B_3E_3 - D_3F_3 dm}{4}\right] = \left(\dfrac{n}{n'}\right)^2 s_1 - n'^2 s_2 - l^2 s_3 \\ 2\dfrac{n}{n'} l \times \left[\dfrac{A_3E_3 - C_3F_3 dm}{4}\right] = \left(\dfrac{n}{n'}\right)^2 s_1 - n'^2 s_2 + l^2 s_3 \\ 2n \times \left[\dfrac{A_3B_3 - C_3D_3 dm}{4}\right] = \left(\dfrac{n}{n'}\right)^2 s_1 + n'^2 s_2 - l^2 s_3 \end{cases}$$

(c) $s_2 = s_3 = s$

2) Let's put $\delta_1 = \text{pgcd}(A_3, C_3)\,, \delta_2 = \text{pgcd}(B_3, D_3)$ et $\delta_3 = \text{pgcd}(E_3, F_3)\,,$ we have:

(a) $\delta_2 = \delta_3 = \delta$ et B_3, E_3, D_3 et F_3 are the same parity.

Lemma 1.3

3) (a)

$$s_3 s_4 l \equiv -\lambda_{n'} s_1 s(d'm') \pmod 4.$$

(b)

$$\begin{aligned}
C_3 &\equiv a_2 a_3 s_1 \times \lambda_{n'} \times \left[\frac{1+s(d'm')}{2}\right] \pmod 2,\\
D_3 &\equiv \left(a_2 a_3 s_1 \frac{n}{2n'} - \frac{a_4 a_5}{2} s_2 l\right) \pmod 4,\\
F_3 &\equiv \left[\frac{a_4 a_5}{2} s_2 n' - a_6 a_7 s_1 s_2 \frac{n}{2n'}\right] \pmod 4,\\
A_3 &\equiv a_2^2 \times m \times \lambda_{n'} \equiv a_2^2 \times d \times \lambda_{n'} \pmod 4.
\end{aligned}$$

Example

Definition:
Let $K_3 = \mathbb{Q}\left(\sqrt{dm}, \sqrt{dn}, \sqrt{d'm'n'l}\right)$ a triquadratic field. We call K_3- integral signal, any 8-upplet of the type $(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7)$ where $\epsilon_i = \pm 1$, such that: $\frac{1}{2^j}(\epsilon_0 + \epsilon_1\sqrt{dm} + \epsilon_2\sqrt{dn} + \epsilon_3\sqrt{mn} + \epsilon_4\sqrt{d'm'n'l} + \epsilon_5\sqrt{\frac{d}{d'}\frac{m}{m'}n'l} + \epsilon_6\sqrt{\frac{d}{d'}\frac{n}{n'}m'l} + \epsilon_7\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}) \in \mathbb{Z}_{K_3}$, and $j \in \{2, 3\}$ depend on the case to which K_3 belongs.

Example

Let's consider case 2; subcase: $(dm, dn, d'm'n'l) \equiv (1, 1, 2) \pmod 4$.

Let's try to build all non trivial integers of:
$K_3 = \mathbb{Q}\left(\sqrt{-42}, \sqrt{2310}, \sqrt{-154}\right)$ of the type:
$\theta = \frac{1}{4}(\epsilon_0 + \epsilon_1\sqrt{33} + \epsilon_2\sqrt{-15} + \epsilon_3\sqrt{-55} + \epsilon_4\sqrt{-154} + \epsilon_5\sqrt{-42} + \epsilon_6\sqrt{2310} + \epsilon_7\sqrt{70})$ where $\epsilon_i = \pm 1$, i.e. find all K_3- integral signals.
First let's show that we are effectively in this subcase mentionned.
To see that, let's do the products modulo squares, we find:
$(33, -15, -55, -154, -42, 2310, 70) \equiv (1, 1, 1, 2, 2, 2, 2) \pmod 4$.

Exmple

So $K_3 = \mathbb{Q}\left(\sqrt{33}, \sqrt{-15}, \sqrt{-154}\right) =$
$\mathbb{Q}\left(\sqrt{3 \times 11}, \sqrt{3 \times (-5)}, \sqrt{1 \times 11 \times 1 \times (2 \times (-7))}\right)$
belongs to case 2.
And so:
$dm = 33, dn = -15, mn = -55, d^{'} m' n' l = -154,$
$\frac{dm}{d'm'} n' l = -42, \frac{dn}{d'n'} m' l = 2310, \frac{mn}{m'n'} d' l = 70.$
Then:
$(d, m, n, d', m', n', l) = (3, 11, -5, 1, 11, 1, -14).$
The signs we need are:
$s_1 = \lambda_d s(d) = -1,\ s_2 = \lambda_{d'm'} s(d'm') = -1,\ s_3 = \lambda_{d'n'} s(d'n') =$
$1, s_4 = \lambda_{dm'n'} s(dm'n') = 1,$ with: $\gamma = \lambda_{d\prime m\prime n\prime \frac{l}{2}} = \lambda_{-77} = -1.$

And the Chatelain's written form is:

$$K_3 = \mathbb{Q}\left(\sqrt{dm},\ \sqrt{dn}, \sqrt{2\lambda_{d'm'n'\frac{l}{2}}}\sqrt{\lambda_{d'm'n'\frac{l}{2}}\ d'm'n'\frac{l}{2}}\right)$$
$$= \mathbb{Q}\left(\sqrt{3 \times 11}, \sqrt{3 \times (-5)}, \sqrt{-2}\sqrt{-(1 \times 11 \times 1 \times (-7))}\right),$$

Example

The Chatelain's β-basis is:

$$\beta = \left\{1, \sqrt{dm}, \sqrt{dn}, s_1\sqrt{mn}, \gamma\sqrt{d'm'n'l}, \gamma s_2\sqrt{\frac{d}{d'}\frac{m}{m'}n'l}, \gamma s_3\sqrt{\frac{d}{d'}\frac{n}{n'}m'l}, \gamma s_4\sqrt{\frac{m}{m'}\frac{n}{n'}d'l}\right\}$$

$$= \left\{1, \sqrt{33}, \sqrt{-15}, -\sqrt{-55}, -\sqrt{-154}, \sqrt{-42}, -\sqrt{2310}, -\sqrt{70}\right\}.$$

Example

Let's come back to our elements θ.

$\theta = \frac{1}{4}(\epsilon_0 + \epsilon_1\sqrt{33} + \epsilon_2\sqrt{-15} + \epsilon_3\sqrt{-55} + \epsilon_4\sqrt{-154} + \epsilon_5\sqrt{-42} + \epsilon_6\sqrt{2310} + \epsilon_7\sqrt{70})$ where $\epsilon_i = \pm 1$, we have:
$\theta = \frac{1}{4}(\epsilon_0 + \epsilon_1\sqrt{33} + \epsilon_2\sqrt{-15} + (-\epsilon_3)(-\sqrt{-55}) + (-\epsilon_4)(-\sqrt{-154}) + \epsilon_5\sqrt{-42} + (-\epsilon_6)(-\sqrt{2310}) + (-\epsilon_7)(-\sqrt{70}))$.
This means that:

$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) = (\epsilon_0, \epsilon_1, \epsilon_2, -\epsilon_3, -\epsilon_4, \epsilon_5, -\epsilon_6, -\epsilon_7)$.
Now θ belongs to $\mathbb{Z}_{K_3}$ if and only if:

Example

$$\begin{cases} a_0 \in \mathbb{Z} \\ a_0 - a_1 \equiv 0 \pmod 2 \\ a_1 - a_2 \equiv 0 \pmod 2 \\ a_0 - a_1 + a_2 - a_3 \equiv 0 \pmod 4 \\ a_4 \in \mathbb{Z} \\ a_4 - a_5 \equiv 0 \pmod 2 \\ a_5 - a_6 \equiv 0 \pmod 2 \\ a_4 - a_5 + a_6 - a_7 \equiv 0 \pmod 4 \end{cases} \Rightarrow$$

$$\begin{cases} \epsilon_0 = \pm 1 \Rightarrow \epsilon_0 \in \mathbb{Z} \\ \epsilon_0 - \epsilon_1 \equiv 0 \pmod 2 \\ \epsilon_1 - \epsilon_2 \equiv 0 \pmod 2 \\ \epsilon_0 - \epsilon_1 + \epsilon_2 + \epsilon_3 \equiv 0 \pmod 4 \\ \epsilon_4 = \pm 1 \Rightarrow \epsilon_4 \in \mathbb{Z} \\ -\epsilon_4 - \epsilon_5 \equiv 0 \pmod 2 \\ \epsilon_5 + \epsilon_6 \equiv 0 \pmod 2 \\ -\epsilon_4 - \epsilon_5 - \epsilon_6 + \epsilon_7 \equiv 0 \pmod 4 \end{cases}$$

Example

$$\Leftrightarrow \begin{cases} \epsilon_0 = \pm 1 \\ \epsilon_1 = \epsilon_0 - 2k_2 \ , \ k_2 \in \{0, \epsilon_0\} \\ \epsilon_2 = \epsilon_0 - 2(k_2 + k_3) \ , \ (k_2, k_3) \in \{\{0\} \times \{0, \epsilon_0\}\} \cup \{\{\epsilon_0\} \times \{0, -\epsilon_0\}\} \\ \epsilon_3 = -\epsilon_0 + 2(k_3 + 2k_4), \\ \ (k_3, k_4) \in \{\{0\} \times \{0\}\} \cup \{\{\epsilon_0\} \times \{0\}\} \cup \{\{-\epsilon_0\} \times \{\epsilon_0\}\} \end{cases}$$

And because the 4 last equations leave the same that the four first ones, we have:

$$(\epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7) = (-\epsilon_0, \epsilon_1, -\epsilon_2, \epsilon_3)$$

Example

$(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3) \in E = \{(\epsilon_1, \epsilon_1, \epsilon_1, -\epsilon_1)\} \cup \{(\epsilon_1, \epsilon_1, -\epsilon_1, \epsilon_1)\} \cup$
$\{(\epsilon_1, -\epsilon_1, -\epsilon_1, -\epsilon_1)\} \cup \{(\epsilon_1, -\epsilon_1, \epsilon_1, \epsilon_1)\}.$
$= \{\pm(-1, 1, 1, 1)\} \cup \{\pm(1, -1, 1, 1)\} \cup$
$\{\pm(1, 1, -1, 1)\} \cup \{\pm(1, 1, 1, -1)\}.$

And then too:

$(\epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7) \in E =$
$\{(-\epsilon_0, \epsilon_0, -\epsilon_0, -\epsilon_0)\} \cup \{(-\epsilon_0, \epsilon_0, \epsilon_0, \epsilon_0)\} \cup \{(-\epsilon_0, -\epsilon_0, \epsilon_0, -\epsilon_0)\} \cup$
$\{(-\epsilon_0, -\epsilon_0, -\epsilon_0, \epsilon_0)\}.$

Example

This means that the solutions $(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7)$ of our problem, are obtained by doing:
$((\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3), (\epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7)) \in E \times E$.

So: $\theta = \frac{1}{4}(\epsilon_0 + \epsilon_1\sqrt{33} + \epsilon_2\sqrt{-15} + \epsilon_3\sqrt{-55} + \epsilon_4\sqrt{-154} + \epsilon_5\sqrt{-42} + \epsilon_6\sqrt{2310} + \epsilon_7\sqrt{70})$ where $\epsilon_i = \pm 1$, belongs to $\mathbb{Z}_{K_3} \Leftrightarrow$
The 64 elements solutions of type 8-upplets
$(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7)$, are such that :
$((\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3), (\epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7)) \in E \times E$, where:
$E = \{\pm(-1, 1, 1, 1)\} \cup \{\pm(1, -1, 1, 1)\} \cup \{\pm(1, 1, -1, 1)\} \cup \{\pm(1, 1, 1, -1)\}$.
Let's note that these 64 solutions can be classified by using the natural action of $Gal(K_3/\mathbb{Q})$ on this same set of solutions.

Then we find finally 8 orbits and each of them has a cardinal equal to 8; given by $Gal(K_3/\mathbb{Q})(\theta_i)$, $i = 0, .., 7$, and each of them is crossing respectively, by only one of the elements :

$$\theta_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} ; \theta_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} ; \theta_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} ; \theta_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

$$; \theta_4 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} ; \theta_5 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} ;\theta_6 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} ; \theta_7 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} .$$

For instance, the orbit S_{θ_0} of θ_0 under $Gal(K_3/\mathbb{Q}) = G$ is the following set:

$$G(\theta_0) = \{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \}.$$

And the irreducible polynomial of θ_0
over $\mathbb{Q}$, whose other roots just above is (from MAPPLE Calculator) :

$Irr(\theta_0, X) = X^8 - 2X^7 - 535X^6 + 1596X^5 + 111509X^4 - 226650X^3 - 11395285X^2 + 677800X + 649147150.$

We get by the same way, the following irreducible polynomials over $\mathbb{Q}$ for $\theta_i, i = 1, ..., 7.$

$Irr(\theta_1, X) = X^8 - 2X^7 - 535X^6 + 1316X^5 + 118789X^4 - 343410X^3 - 11884725X^2 + 16299000X + 584808750;$

$Irr(\theta_2, X) = X^8 - 2X^7 - 535X^6 - 840X^5 + 118019X^4 + 419814X^3 - 12451753X^2 - 33307172X + 568782124;$

$Irr(\theta_3, X) = X^8 - 2X^7 - 535X^6 + 980X^5 + 105349X^4 - 350466X^3 - 10605573X^2 + 49155768X + 676247454;$

$Irr(\theta_4, X) = X^8 + 2X^7 - 535X^6 - 1596X^5 + 111509X^4 + 226650X^3 - 11395285X^2 - 677800X + 649147150;$

$Irr(\theta_5, X) = X^8 + 2X^7 - 535X^6 - 1316X^5 + 118789X^4 + 343410X^3 - 11884725X^2 - 16299000X + 584808750;$

$Irr(\theta_6, X) = X^8 + 2X^7 - 535X^6 + 840X^5 + 118019X^4 - 419814X^3 - 12451753X^2 + 33307172X + 568782124;$

$Irr(\theta_7, X) = X^8 + 2X^7 - 535X^6 - 980X^5 + 105349X^4 + 350466X^3 - 10605573X^2 - 49155768X + 676247454.$

[1] D. Chatelain: Bases des entiers des corps composés par des extensions quadratiques de $\mathbb{Q}$, Ann . Sci. Univ. Besançon Math,. Fasc. 6, 38 pp (1973).

[2] M.-N. Gras, F. E. Tanoé: Corps biquadratiques monog ènes, manuscripta Mathematica, 86, 63-75 (1995).

[3] B. He, A. Togbé: Simultaneous Pellian Equation with a Single or no Solution, Acta Arith. 134 , 369-380 (2008).

[4] K. V. Kouakou: Monogénéité des Corps Triquadratiques, Thèse unique, Université Félix Houphouët BOIGNY, UFRMI, Labo Math-Fondamentales, N° d'ordre: 2044/2017,(août 2017).

[5] Kouassi V. KOUAKOU & François E. Tanoé: Chatelain's integral bases for Triquadratic Number Fields, Afr. Mat., (Mars 2017), Issue 1-2, 1-31, DOI 10.1007/s13370-016-0428-x.

[6] Y. Motoda: Note on quartic fields, Rep.Fac. Sci. Engrg. Saga Uni.Math.,Vol. 32, N° 1, 19 pp (2003).

[7] Y. Motoda; T. Nakahara: Power integral bases in algebraic number fields whose Galois groups are 2-elementary abelian, Arch. Math. 83 (2004) 309–316, 0003–889X/04/040309–08; DOI 10.1007/s00013-004-1077-0, © Birkhauser Verlag, bases, 2004 Archiv der Mathematik.

[8] Y. Motoda; T. Nakahara ; K. H. Park: On power integral bases of the 2-elemenatry abelian extensioon fields; Trends in Mathematics, Information Center for Mathematical Sciences, Volume 9, Number 1, June, 2006, Pages 55-63.

[9] G. Nyul: Non-monogenity of multiquadratic number fields, Acta Mathematica & Informatica Universitatis Ostraviensis, Vol. 10 , N ° 1, 85-93 (2002).

[10] K.H.Park; Y. Motoda; T. Nakahara: On Integral bases of Certain Real Octic Abelian Fields, Rep.Fac. Sci. Engrg. Saga Uni.Math.,Vol. 34, N ° 1, 15 pp (2005).

[11] K. H. Park ;T. Nakahara ;Y. Motoda: On integral bases of real octic 2-elementary abelian extensions, Kyoto University Research Information Repository Departement Bulletin Paper (2006) 1521:174-184.

[12] François. E. Tanoé & Vincent Kouassi: Generators of power integral bases of $\mathbb{Q}(\zeta_{24}) = \mathbb{Q}(\sqrt{-3}, \sqrt{2}, \sqrt{-1})$; Annales Mathématiques Africaines, Volume 5 (2015) pp. 117-131. ISSN print 2305-3836 ISSN online 2218-4414.

[13] François. E. Tanoé & Kouassi Vincent Kouakou: Characterization of integers of triquadratic Number Fields; Fundamental Journal of Mathematics and Mathematical Sciences, Volume 14, Issue 2, 2020, Pages 49-74 p-ISSN:2395-7573;e-ISSN:2395-7581 This paper is available online at http://www.frdint.com/

[14] François. E. Tanoé & Vincent Kouassi Kouakou: Diophantine proof of non-monogeneity for triquadratic Number Fields with odd discriminant; Fundamental Journal of Mathematics and

Mathematical Sciences, This paper will be available online at http://www.frdint.com/ to be published

[15] F. E. Tanoé: Chatelain's Integer bases for biquadratic fields, Afr. Mat. (2017) 28:727–744; DOI 10.1007/s13370-016-0476-2. Received: 2 February 2015 / Accepted: 19 December 2016 / Published online: 20 January 2017 , African Mathematical Union and Springer-Verlag Berlin Heidelberg 2017.

[16] F. E. Tanoé: Proof of a Monogenesis Conjecture Involving one Unit in Biquadratic Number fields, Proc. Fifth Int. Workshop on Contempory Problems in Mathematical Physics ,Cotonou, Bénin, oct – nov 2007, 292-298, J. Govaerts, M.N. Hounkonnou, eds (.International Chair in Mathematical physics and Applications ICMPA-UNESCO CHAIR, University of Abomey-Calavi, Republic of BENIN, December 2008).