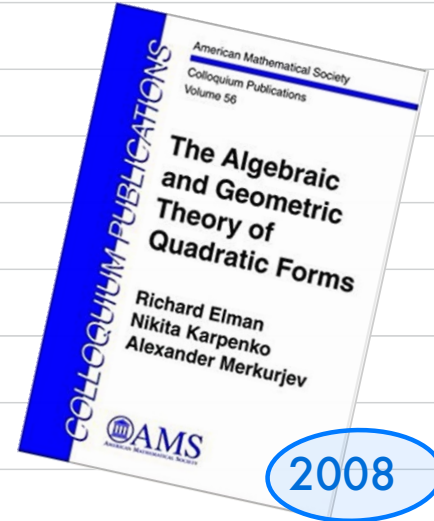
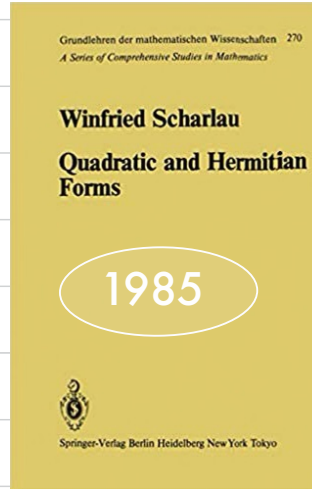


Quadratic Forms : from number theory to algebra and geometry



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What is a quadratic form ?

F field,

A quadratic form over F

is $\varphi: V \rightarrow F$

F -vector-space

$\dim_F V = n = \dim(\varphi)$

such that

① $\varphi(v\lambda) = \varphi(v)\lambda^2 \quad \forall v \in V, \forall \lambda \in F$

② $B_\varphi: V \times V \rightarrow F$

$$(v, w) \mapsto \varphi(v+w) - \varphi(v) - \varphi(w)$$

is a (symmetric) bilinear form

Exple

$$\varphi: F^n \rightarrow F, \quad v = (x_1, \dots, x_n) \mapsto \sum a_i x_i^2, \quad a_i \in F$$

$$B_\varphi(v, v') = 2 \sum_{i=1}^n a_i x_i x'_i = v \begin{pmatrix} 2a_1 & 0 \\ 0 & 2a_n \end{pmatrix} v'^t$$

φ is called diagonal, $\varphi = \langle a_1, \dots, a_n \rangle$

More examples / Diagonalization

K/F quadratic separable, $N_{K/F} K \rightarrow F$ is a quad form
clar $F \neq 2$, $K = F[t]/(t^2 - a)$, $N_{K/F}(x, y) = x^2 - ay^2$ $\langle 1, -a \rangle$
clar $F = 2$, $K = F[t]/(t^2 + t + a)$, $N_{K/F}(x, y) = x^2 + xy + ay^2$ $[1, a]$

From now on clar $F \neq 2$

Fact All quadratic forms are diagonalizable

In a suitable basis (orthogonal basis)

$$q(v) = \sum a_i x_i^2, \quad v = (x_1, \dots, x_n)$$

Expe $(x, y) \mapsto x^2 + xy = x(x+y) = x'y' = \left(\frac{x'+y'}{2}\right)^2 - \left(\frac{x'-y'}{2}\right)^2$
 $H = [1, 0] \cong \langle 1, -1 \rangle \leftarrow \text{Hyperbolic plane}$

Non degenerate forms / Isometry

We only consider non degenerate forms

$$\underline{\text{ie}} \quad \hat{B}_\varphi \quad V \longrightarrow V^* \\ v \longmapsto (w \mapsto B_\varphi(v, w)) \quad \Big) \text{ is an isomorphism}$$

Exple Euclidean space - $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$
 $q = \langle a_1, \dots, a_n \rangle, \quad n = \dim_F V, \quad a_i \in F^\times$

Def An isometry $f: (V, \varphi) \rightarrow (W, \psi)$ is an F -linear isomorphism $f: V \rightarrow W$ such that $\forall v \in V, \quad \psi(f(v)) = \varphi(v)$

Exple $(x, y) \mapsto x^2 + xy = x(x+y) = x'y' = \left(\frac{x'+y'}{2}\right)^2 - \left(\frac{x'-y'}{2}\right)^2$
 $H = [1, 0] \cong \langle 1, -1 \rangle \leftarrow \text{Hyperbolic plane}$

Example of classical questions

- ① Isotropy question Given φ/F
 $\exists v \in V, v \neq 0 \text{ s.t. } \varphi(v) = 0$?
(If so, φ is called isotropic)
- ② More generally Describe the set of values of φ
 $D_F(\varphi) = \{\varphi(v), v \in V, v \neq 0\}$
- ③ Classification question Classify quadratic forms
over a given field F up to isometry

Classification question

$$F = \mathbb{C}$$

$$\varphi(x_1, \dots, x_n) = \sum a_i x_i^2 = \sum (\alpha_i x_i)^2 \quad \left| \quad \varphi \cong \langle 1, \dots, 1 \rangle \right.$$
$$\alpha_i \in \mathbb{C}, \alpha_i^2 = a_i$$

$$\left[\varphi \cong_{\mathbb{C}} \psi \iff \dim \varphi = \dim \psi \right]$$

dim } Isometry classes of quad forms / $\mathbb{C}$ } $\xrightarrow{\text{bijection}}$ $\mathbb{N}$

$$F = \mathbb{R} \quad \varphi = \langle a_1, \dots, a_n \rangle \cong \underbrace{\langle 1, \dots, 1 \rangle}_r, \underbrace{\langle -1, \dots, -1 \rangle}_s$$

$$r = \# \{a_i, a_i > 0\}$$

$$s = \# \{a_i, a_i < 0\} \quad (r, s) \stackrel{\text{def}}{=} \text{signature of } \varphi$$

$$\left[\varphi \cong_{\mathbb{R}} \psi \iff \text{they have the same signature} \right]$$

The Hasse-Minkowski theorem

Scharlau : << For a long time - at least from Fermat to Minkowski - the theory of quadratic forms was a part of number theory. >>

$$F = \mathbb{Q}$$

$$\varphi \quad v \rightarrow \mathbb{Q}$$

quadratic form

For any place v of $\mathbb{Q}$,

$$\varphi_v \quad v \otimes_{\mathbb{Q}} \mathbb{Q}_v \rightarrow \mathbb{Q}_v$$

Thm (Hasse-Minkowski)

$$\lfloor \varphi \text{ is isotropic} \Leftrightarrow \varphi_v \text{ is isotropic } \forall v$$

$$\text{Cor } \lfloor \varphi \underset{\mathbb{Q}}{\cong} \psi \Leftrightarrow \varphi_v \underset{\mathbb{Q}_v}{\cong} \psi_v \quad \forall v$$

The algebraic theory of quadratic forms

Scharlau : << Witt opened in 1937 a new chapter in the theory of quadratic forms .../... Thirty years later, Pfister demonstrated the significance of this approach by his celebrated structure theorems. >>

- ① Consider quadratic forms over arbitrary fields
- ② { All quadratic forms over F } $\leadsto W(F)$ Witt ring

Cassels' and Pfister's Subform theorem

$$\varphi: V \rightarrow F, \quad \psi: W \rightarrow F$$

Def ψ is a subform of φ ($\psi \subset \varphi$) if there exists
an injective F -linear map $f: W \hookrightarrow V$ s.t. $\forall w \in W$,
 $\varphi(f(w)) = \psi(w)$

(If no, there exists (w', ψ') , $V \cong W \oplus W'$, $\varphi(w+w') = \psi(w) + \psi'(w')$)

If $\psi = \langle a_1, \dots, a_r \rangle$, then $\varphi = \langle a_1, \dots, a_r, a_{r+1}, \dots, a_n \rangle$

Thm Assume φ is anisotropic and $\psi = \langle a_1, \dots, a_r \rangle$

Then $\psi \subset \varphi \iff \varphi_{\underbrace{F(t_1, \dots, t_r)}_{\text{variables}}}$ represents $a_1 t_1^2 + \dots + a_r t_r^2$

The Witt ring of F

Witt ring $W(F) = \{ \text{Quadratic forms over } F \} / \{ \text{Hyperbolic Forms} \}$

Operations $\perp, \otimes$

$IF \subset W(F)$

" ideal of Witt classes of even dimensional quad forms

Additively generated by $\langle 1, -a \rangle$ ($\langle a, b \rangle \sim \langle 1, -1, a, b \rangle$
 $\sim \langle 1, a \rangle - \langle 1, -b \rangle$)

— generated by $\langle a_1, \dots, a_n \rangle$

$$W(F) \supset IF \supset I^2 F \supset \dots \supset I^n F \supset \bigoplus_{i=1}^n \langle 1, -a_i \rangle$$

n fold - Pfister forms

Milnor's conjecture (Theorem of Orlov-Vishik-Voevodsky)

There is a commutative triangle of isomorphisms

$$\begin{array}{ccc} & G \otimes_* F = \bigoplus_{n \geq 0} I^n F / I^{n+1} F & \\ \nearrow \cong & \downarrow \parallel & \\ k_*^{\Gamma}(F) & & H^*(F) = \bigoplus_{n \geq 0} H^n(Gal(\bar{F}/F) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \\ \searrow \cong & & \end{array}$$

Milnor's K-theory
modulo 2

Classification of quadratic forms over F

$$W(F) \supset I(F) \supset I^2(F) \supset \dots \supset I^n(F) \supset \dots \quad \Bigg| \quad \bigcap_{n \geq 0} I^n(F) = \{0\} \subset W(F)$$

φ, ψ quadratic forms / F of the same dimension

$\varphi \cong \psi \iff \varphi \perp (-\psi)$ is hyperbolic form

$$\iff e_1(\varphi \perp (-\psi)) = 0 \quad \forall.$$

Pfister Hauptsatz A non-trivial form in $I^n(F)$ has
dimension $\geq 2^n$

Motivic equivalence of quadratic forms

$$\varphi / F \quad \varphi = \langle a_1, \dots, a_n \rangle$$

X_φ = projective quadric with equation $a_1 x_1^2 + \dots + a_n x_n^2 = 0$

mod $\Gamma(X_\varphi)$ = motif of X_φ
(Chow motives with coefficients in $\mathbb{F}_2$)

Facts $\left[\begin{array}{l} \varphi \text{ is isotropic iff } H \subset \varphi \\ \varphi = \bigoplus H \perp \varphi_{an} \end{array} \right.$ anisotropic
with index
 $m = i_w(\varphi)$

Thm $(V, \varphi, \psi / F)$

$\left[\Gamma(X_\varphi) \cong \Gamma(X_\psi) \iff \forall E/F, i_w(\varphi_E) \cong i_w(\psi_E) \right.$
in the category of Chow-motives