

A short introduction to modular forms

Cécile Armana

Université Bourgogne Franche-Comté, Besançon, France

Research School in Number Theory and Applications

IMSP Benin – December 13, 2021



Outline

1. **Our motivation: sums of four squares**
2. **Definitions of modular forms**
3. **Examples!**
4. **Dimensions of spaces**
5. **Arithmetical identities**
6. **Back to sums of four squares**

Outline

1. Our motivation: sums of four squares
2. Definitions of modular forms
3. Examples!
4. Dimensions of spaces
5. Arithmetical identities
6. Back to sums of four squares

References:

- J.-P. Serre, *Cours d'arithmétique/A course in arithmetic*
<https://www.math.purdue.edu/~jlipman/MA598/Serre-Course%20in%20Arithmetic.pdf>
- D. Zagier, *The 1-2-3 of Modular forms*
https://people.mpim-bonn.mpg.de/zagier/files/doi/10.1007/978-3-540-74119-0_1/fulltext.pdf



Our motivation: sums of four squares

Lagrange (1770). Every natural number can be represented as the sum of four integer squares.

Our motivation: sums of four squares

Lagrange (1770). Every natural number can be represented as the sum of four integer squares.

Number of representations:

$$r(n) = \#\{(a, b, c, d) \in \mathbb{Z}^4 \mid n = a^2 + b^2 + c^2 + d^2\}$$

Examples :

- $n = 2 : (\pm 1)^2 + (\pm 1)^2 + 0 + 0, (\pm 1)^2 + 0 + (\pm 1)^2 + 0, \dots$
 $r(2) = 24$
- $n = 5 : (\pm 2)^2 + (\pm 1)^2 + 0^2 + 0^2, \dots$
 $r(5) = 48$

Our motivation: sums of four squares

Lagrange (1770). Every natural number can be represented as the sum of four integer squares.

Number of representations:

$$r(n) = \#\{(a, b, c, d) \in \mathbb{Z}^4 \mid n = a^2 + b^2 + c^2 + d^2\}$$

Examples :

- $n = 2 : (\pm 1)^2 + (\pm 1)^2 + 0 + 0, (\pm 1)^2 + 0 + (\pm 1)^2 + 0, \dots$
 $r(2) = 24$
- $n = 5 : (\pm 2)^2 + (\pm 1)^2 + 0^2 + 0^2, \dots$
 $r(5) = 48$

Jacobi's four-square theorem (1829)

For any $n \in \mathbb{N}$, $r(n) = 8 \sum_{d|n, d \not\equiv 0 \pmod{4}} d$.

Proof: modular forms, theta functions... the goal of this talk!

The group $\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \mid ad - bc = 1 \right\}$

The upper-half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$

Let $k \in \mathbb{Z}$.

The group $\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \mid ad - bc = 1 \right\}$

The upper-half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$

Let $k \in \mathbb{Z}$.

Definition

A **modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$** is a holomorphic function $f : \mathbb{H} \rightarrow \mathbb{C}$ such that:

- (i) for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathbb{H}$, $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$
- (ii) there exists $(c_n)_n \in \mathbb{C}$ such that $f(z) = \sum_{n \geq 0} c_n q^n$ where $q = e^{2i\pi z}$.

The group $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \mid ad - bc = 1 \right\}$

The upper-half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$

Let $k \in \mathbb{Z}$.

Definition

A **modular form of weight k for $SL_2(\mathbb{Z})$** is a holomorphic function $f : \mathbb{H} \rightarrow \mathbb{C}$ such that:

- (i) for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $z \in \mathbb{H}$, $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$
- (ii) there exists $(c_n)_n \in \mathbb{C}$ such that $f(z) = \sum_{n \geq 0} c_n q^n$ where $q = e^{2i\pi z}$.

Condition (i) with $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ gives $f(z+1) = f(z)$, **1-periodicity**.

$\Rightarrow \exists (c_n)_n$ such that $f(z) = \sum_{n \in \mathbb{Z}} c_n q^n$. Condition (ii) is stronger (**holomorphy at $i\infty$**).

About (i): $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \forall z \in \mathbb{H}, f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$

“Symmetries”, “family of functional equations” with respect to the group $SL_2(\mathbb{Z})$. Since $SL_2(\mathbb{Z})$ is generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$:

$$(i) \iff f(z+1) = f(z), f\left(-\frac{1}{z}\right) = z^k f(z)$$

About (i): $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \forall z \in \mathbb{H}, f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$

“Symmetries”, “family of functional equations” with respect to the group $SL_2(\mathbb{Z})$. Since $SL_2(\mathbb{Z})$ is generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$:

$$(i) \iff f(z+1) = f(z), f\left(-\frac{1}{z}\right) = z^k f(z)$$

About (ii): $f(z) = \sum_{n \geq 0} c_n q^n$

Called the **Fourier expansion** (or q -expansion) of f .

General philosophy of modular forms:

The Fourier coefficients c_n should carry interesting arithmetic information.

About (i): $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \forall z \in \mathbb{H}, f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$

“Symmetries”, “family of functional equations” with respect to the group $SL_2(\mathbb{Z})$. Since $SL_2(\mathbb{Z})$ is generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$:

$$(i) \iff f(z+1) = f(z), f\left(-\frac{1}{z}\right) = z^k f(z)$$

About (ii): $f(z) = \sum_{n \geq 0} c_n q^n$

Called the **Fourier expansion** (or q -expansion) of f .

General philosophy of modular forms:

The Fourier coefficients c_n should carry interesting arithmetic information.

Existence of non-zero modular forms?

Spaces of modular forms

The space M_k of modular forms of weight k is a $\mathbb{C}$ -vector space.

If $f \in M_k$ and $g \in M_{k'}$ then $fg \in M_{k+k'}$.

$\bigoplus_{k \in \mathbb{Z}} M_k$ (with multiplication) is a $\mathbb{C}$ -algebra graded by the weight.

Spaces of modular forms

The space M_k of modular forms of weight k is a $\mathbb{C}$ -vector space.

If $f \in M_k$ and $g \in M_{k'}$ then $fg \in M_{k+k'}$.

$\bigoplus_{k \in \mathbb{Z}} M_k$ (with multiplication) is a $\mathbb{C}$ -algebra graded by the weight.

Cuspidal modular forms

A modular form f is called **cuspidal** if $c_0 = 0$ i.e. $f(z) = \sum_{n \geq 1} c_n q^n$.

The set S_k of cuspidal forms is a subspace of M_k (and $\dim M_k/S_k \leq 1$).

Spaces of modular forms

The space M_k of modular forms of weight k is a $\mathbb{C}$ -vector space.

If $f \in M_k$ and $g \in M_{k'}$ then $fg \in M_{k+k'}$.

$\bigoplus_{k \in \mathbb{Z}} M_k$ (with multiplication) is a $\mathbb{C}$ -algebra graded by the weight.

Cuspidal modular forms

A modular form f is called **cuspidal** if $c_0 = 0$ i.e. $f(z) = \sum_{n \geq 1} c_n q^n$.

The set S_k of cuspidal forms is a subspace of M_k (and $\dim M_k/S_k \leq 1$).

The case of odd weight

Using $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ in (i), we get $f(z) = (-1)^k f(z)$.

Hence if k is odd, $M_k = S_k = \{0\}$.

Let $k \geq 4$ be an even integer.

Eisenstein series

For $z \in \mathbb{H}$: $G_k(z) = \frac{(k-1)!}{2(2\pi i)^k} \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z} - \{(0,0)\}} \frac{1}{(mz + n)^k}$ (converges).

Let $k \geq 4$ be an even integer.

Eisenstein series

$$\text{For } z \in \mathbb{H}: G_k(z) = \frac{(k-1)!}{2(2\pi i)^k} \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z} - \{(0,0)\}} \frac{1}{(mz + n)^k} \quad (\text{converges}).$$

Fourier expansion:

$$G_k(z) = \frac{(-1)^{k/2}(k-1)!}{(2\pi)^k} \zeta(k) + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

$$\text{where } \sigma_h(n) = \sum_{d>0, d|n} d^h \text{ (sums of powers of divisors) and } \zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$$

(Riemann zeta function). Hence $G_k \in M_k$ (and $G_k \notin S_k$).

Let $k \geq 4$ be an even integer.

Eisenstein series

$$\text{For } z \in \mathbb{H}: G_k(z) = \frac{(k-1)!}{2(2\pi i)^k} \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z} - \{(0,0)\}} \frac{1}{(mz + n)^k} \quad (\text{converges}).$$

Fourier expansion:

$$G_k(z) = \frac{(-1)^{k/2}(k-1)!}{(2\pi)^k} \zeta(k) + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

$$\text{where } \sigma_h(n) = \sum_{d>0, d|n} d^h \text{ (sums of powers of divisors) and } \zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$$

(Riemann zeta function). Hence $G_k \in M_k$ (and $G_k \notin S_k$).

Remark: The constant coefficient is also $-\frac{B_k}{2k}$ where $B_k \in \mathbb{Q}$ is the k th Bernoulli number. All Fourier coefficients of G_k are in $\mathbb{Q}$.

First coefficients of Eisenstein series

$$G_4(z) = \frac{1}{240} + q + 9q^2 + 28q^3 + 73q^4 + 126q^5 + 252q^6 + \dots$$

$$G_6(z) = \frac{-1}{504} + q + 33q^2 + 244q^3 + 1057q^4 + \dots$$

$$G_8(z) = \frac{1}{480} + q + 129q^2 + 2188q^3 + \dots$$

$$G_{10}(z) = \frac{-1}{264} + q + 513q^2 + \dots$$

$$G_{12}(z) = \frac{691}{65520} + q + 2049q^3 + \dots$$

Now a cuspidal example:

Definition (The Delta function)

$$\Delta(z) = 8000G_4^3(z) - 147G_6^2(z) \in M_{12}$$

Now a cuspidal example:

Definition (The Delta function)

$$\Delta(z) = 8000G_4^3(z) - 147G_6^2(z) \in M_{12}$$

Fourier expansion:

$$\Delta(z) = q - 24q^2 + 252q^3 - 1472q^4 + \dots$$

Hence $\Delta \in S_{12}$.

(the coefficients 8000 and -147 were chosen to make this happen!)

Dimensions of spaces

Theorem

The vector spaces M_k and S_k have *finite dimension* over $\mathbb{C}$.

The proof uses:

- For any even $k \geq 4$, $M_k = \langle G_k \rangle \oplus S_k$.
- The map $f \mapsto \frac{f}{\Delta}$ gives an isomorphism $S_k \simeq M_{k-12}$.
- If $f \in M_k \setminus \{0\}$, there exists integers $n, n', n'', m \geq 0$ such that:

$$n + \frac{1}{2}n' + \frac{1}{3}n'' + m = \frac{k}{12}.$$

Using these facts we also get:

k even	< 0	0	2	4	6	8	10	12	$\dots$	k	$\dots$	$k + 12$	$\dots$
$\dim M_k$	0	1	0	1	1	1	1	2	$\dots$	d	$\dots$	$d + 1$	$\dots$

Exercise. Recover the values in the table.

Principle

Let $d = \dim M_k$ and $\ell > d$. Any family of ℓ modular forms in M_k is linearly dependent over $\mathbb{C}$.

→ Write down the linear relation between them.

→ Uniqueness of Fourier expansions gives identities between the coefficients.

Example. G_4^2, G_8 in M_8 and $\dim M_8 = 1$. There exists $\lambda \in \mathbb{C}$ such that $G_8 = \lambda G_4^2$. Looking at the constant coefficients, we get $G_8 = 120 G_4^2$.

Uniqueness of the Fourier expansion now gives:

$$\forall n \geq 0, \quad \sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m) \sigma_3(n-m)$$

where $\sigma_h(n) = \sum_{d>0, d|n} d^h$ (sums of powers of divisors).

Exercise. Using $\dim M_{10} = 1$, find another arithmetical identity!

Let's go back to the modular form $\Delta \in S_{12}$:

$$\begin{aligned}\Delta &= 8000G_4^3 - 147G_6^2 = \sum_{n=1}^{+\infty} \tau(n)q^n && \text{(Ramanujan tau function)} \\ &= q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 + \dots \\ &= q \prod_{n=1}^{\infty} (1 - q^n)^{24} && \text{(Jacobi infinite product formula).}\end{aligned}$$

We have $\tau(n) \in \mathbb{Z}$.

A congruence of Ramanujan

$$\forall n \geq 0, \quad \tau(n) \equiv \sigma_{11}(n) \pmod{691}$$

A congruence of Ramanujan

$$\forall n \geq 0, \quad \tau(n) \equiv \sigma_{11}(n) \pmod{691}$$

Proof. $\dim M_{12} = 2$ so $\dim S_{12} = 1$ and $S_{12} = \mathbb{C}\Delta$.

$$G_{12} = \frac{691}{65520} + \cdots \quad G_6 = \frac{1}{504} + \cdots$$

We consider $65520G_{12} - 691(504)^2G_6^2 \in S_{12}$. There exists $\alpha \in \mathbb{C}$ with

$$65520G_{12} - 691(504)^2G_6^2 = \alpha\Delta$$

In fact $\alpha \in \mathbb{Z}$. Look at the Fourier coefficients and reduce modulo 691:

$$65520\sigma_{11}(n) \equiv \alpha\tau(n) \pmod{691}$$

Take $n = 1$ gives $\alpha \equiv 65520 \equiv 566 \pmod{691}$ which is invertible modulo 691.

We get:

$$\sigma_{11}(n) \equiv \tau(n) \pmod{691}$$

□

On the Ramanujan tau function

Multiplicativity

If $\gcd(m, n) = 1$, $\tau(mn) = \tau(m)\tau(n)$.

Conjectured by Ramanujan (1916), proved by Mordell (1917). Consequence of the [theory of Hecke operators](#) on modular forms.

Growth of coefficients

For all primes p , $\tau(p) \leq 2p^{11/2}$.

Conjectured by Ramanujan (1916), proved by Deligne (1969) as a consequence of his [proof of the Weil conjectures for algebraic varieties over finite fields](#).

Non-vanishing

Conjecture of Lehmer: for all $n \geq 1$, $\tau(n) \neq 0$.

Unproved! Checked for all $n \leq 816212624008487344127999$ (Derickx, van Hoeij, Zeng, 2013).

The generating function of $r(n)$, the number of representations of n as a sum of four squares:

$$\sum_{n \geq 0} r(n) X^n = \sum_{(a,b,c,d) \in \mathbb{Z}^4} X^{a^2+b^2+c^2+d^2} = \left(\sum_{n \in \mathbb{Z}} X^{n^2} \right)^4$$

Theta function

$$\begin{aligned} \Theta : \mathbb{H} &\longrightarrow \mathbb{C} \\ z &\longmapsto \sum_{n \in \mathbb{Z}} q^{n^2} \quad \text{where } q = e^{2i\pi z} \end{aligned}$$

The generating function of $r(n)$, the number of representations of n as a sum of four squares:

$$\sum_{n \geq 0} r(n) X^n = \sum_{(a,b,c,d) \in \mathbb{Z}^4} X^{a^2+b^2+c^2+d^2} = \left(\sum_{n \in \mathbb{Z}} X^{n^2} \right)^4$$

Theta function

$$\begin{aligned} \Theta : \mathbb{H} &\longrightarrow \mathbb{C} \\ z &\longmapsto \sum_{n \in \mathbb{Z}} q^{n^2} \quad \text{where } q = e^{2i\pi z} \end{aligned}$$

Using Poisson summation formula:

Transformation property for Θ^4

$$\Theta^4(z+1) = \Theta^4(z) \quad \text{and} \quad \Theta^4\left(\frac{z}{4z+1}\right) = (4z+1)^2 \Theta^4(z)$$

Need to generalize the notion of modular form for $\mathrm{SL}_2(\mathbb{Z})$.

$$N \geq 1, \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}, \text{ subgroup of } \mathrm{SL}_2(\mathbb{Z})$$

Definition

A modular form of weight k for $\Gamma_0(N)$ is a holom. function $f : \mathbb{H} \rightarrow \mathbb{C}$ s.t.:

- (i) for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and $z \in \mathbb{H}$, $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$
- (ii) for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, the map $z \mapsto (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$ has a Fourier expansion of the form $\sum_{n \geq 0} c_n(\gamma) q_N^n$ where $q_N = e^{2i\pi z/N}$.

$$N \geq 1, \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}, \text{ subgroup of } \mathrm{SL}_2(\mathbb{Z})$$

Definition

A **modular form of weight k for $\Gamma_0(N)$** is a holom. function $f : \mathbb{H} \rightarrow \mathbb{C}$ s.t.:

- (i) for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and $z \in \mathbb{H}$, $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$
- (ii) for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, the map $z \mapsto (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$ has a Fourier expansion of the form $\sum_{n \geq 0} c_n(\gamma) q_N^n$ where $q_N = e^{2i\pi z/N}$.

They form $\mathbb{C}$ -vector spaces $M_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N))$ (for cuspidal forms), of **finite dimension**. For $N = 1$, we recover M_k .

$\Theta^4 \in M_2(\Gamma_0(4))$

Let's go back to Eisenstein series. The sum

$$G_k(z) = \frac{(k-1)!}{2(2\pi i)^k} \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z} - \{(0,0)\}} \frac{1}{(mz+n)^k} \text{ does not converge for } k=2.$$

False Eisenstein series of weight 2

$$G_2^*(z) = \lim_{s \rightarrow 0} \frac{1}{2(2i\pi)^2} \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z} - \{(0,0)\}} \frac{1}{(mz+n)^2} \frac{1}{|mz+n|^s}, \text{ convergent.}$$

Let's go back to Eisenstein series. The sum

$$G_k(z) = \frac{(k-1)!}{2(2\pi i)^k} \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z} - \{(0,0)\}} \frac{1}{(mz+n)^k} \text{ does not converge for } k=2.$$

False Eisenstein series of weight 2

$$G_2^*(z) = \lim_{s \rightarrow 0} \frac{1}{2(2i\pi)^2} \sum_{(m,n) \in \mathbb{Z} \times \mathbb{Z} - \{(0,0)\}} \frac{1}{(mz+n)^2} \frac{1}{|mz+n|^s}, \text{ convergent.}$$

Satisfies property (i) for $SL_2(\mathbb{Z})$ but is not holomorphic:

$$G_2^*(z) = \frac{1}{8\pi y} - \frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n) q^n \quad \text{where } y = \text{Im } z.$$

(Recall: $M_2 = \{0\}$) We can get rid of the term $\frac{1}{8\pi y}$:

$$z \mapsto 4G_2^*(4z) - G_2^*(z) \in M_2(\Gamma_0(4))$$

$4G_2^*(4z) - G_2^*(z)$ and Θ^4 both belong to $M_2(\Gamma_0(4))$

$\dim M_2(\Gamma_0(4)) = 2\dots$ but $\dim S_2(\Gamma_0(4)) = 0$

$4G_2^*(4z) - G_2^*(z)$ and Θ^4 both belong to $M_2(\Gamma_0(4))$

$\dim M_2(\Gamma_0(4)) = 2 \dots$ but $\dim S_2(\Gamma_0(4)) = 0$

Look at the Fourier expansions:

$$\Theta^4(z) = 1 + \dots, \quad 4G_2^*(4z) - G_2^*(z) = -\frac{3}{24} + \dots$$

Therefore:

$$4G_2^*(4z) - G_2^*(z) + \frac{3}{24}\Theta^4(z) = q + \dots \in S_2(\Gamma_0(4))$$

We get the identity:

$$4G_2^*(4z) - G_2^*(z) = -\frac{3}{24}\Theta^4(z)$$

From:

$$4G_2^*(4z) - G_2^*(z) = -\frac{3}{24}\Theta^4(z)$$

identify their Fourier expansions:

$$1 + 8 \sum_{n=1}^{\infty} \left(\sum_{d|n, d \not\equiv 0 \pmod{4}} d \right) q^n = \left(\sum_{n \in \mathbb{Z}} q^{n^2} \right)^4 = \sum_{n \geq 0} r(n) q^n$$

so for any $n > 0$:

$$r(n) = 8 \sum_{\substack{d|n \\ d \not\equiv 0 \pmod{4}}} d.$$

This proves Jacobi's four-square theorem.

From:

$$4G_2^*(4z) - G_2^*(z) = -\frac{3}{24}\Theta^4(z)$$

identify their Fourier expansions:

$$1 + 8 \sum_{n=1}^{\infty} \left(\sum_{d|n, d \not\equiv 0 \pmod{4}} d \right) q^n = \left(\sum_{n \in \mathbb{Z}} q^{n^2} \right)^4 = \sum_{n \geq 0} r(n) q^n$$

so for any $n > 0$:

$$r(n) = 8 \sum_{\substack{d|n \\ d \not\equiv 0 \pmod{4}}} d.$$

This proves Jacobi's four-square theorem.

Thank you for your attention!